Comp 411
Principles of Programming Languages
Lecture 7
Meta-interpreters

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Denotational Semantics

- The primary alternative to *syntactic* semantics is *denotational* semantics. A denotational semantics maps abstract syntax trees into a set of *denotations* (mathematical values like numbers, lists, and functions).
- The denotations of simple data values like numbers and lists are essentially the same mathematical objects as syntactic values: they have simple inductive definitions with exactly the same structure as the corresponding abstract syntax trees.
- But denotations can also be complex mathematical objects like *functions* or *sets*. For example, the denotation for a lambda-abstraction in "pure" (functional) Scheme is a function mapping denotations to denotations--*not* some syntax tree as in a syntactic semantics.

Meta-interpreters

- Denotational semantics is rooted in mathematical logic: the semantics of terms (expressions) in the predicate calculus is defined denotationally by *recursion* on the syntactic structure of terms. The meaning of each term is a value in a mathematical *structure* or algebra.
- In the realm of programming languages, purely functional interpreters (defined by recursion on the structure of ASTs) constitute a restricted form of denotational definition.
  - The defect is that the output of an actual interpreter is restricted to values that can be characterized syntactically. (How do you output a function?)
  - On the other hand, interpreters naturally introduce a simple form of functional abstraction. An efficient recursive interpreter accepts an extra input: an *environment* mapping free variables to values, thus defining the meaning of a program expression as a function from environments to values.
  - Syntactic interpreters are *not denotational* because they transform ASTs. A denotational interpreter uses pure structural recursion. To handle the bindings to variables, it cannot perform substitutions; it must maintain an environment of bindings instead.

Meta-interpreters cont.

- Interpreters written in a denotational style are often called *meta*-interpreters because they are defined in a meta-mathematical framework where programming language expressions and denotations are objects in the framework. The definition of the interpreter is a level above definitions of functions in the language being defined.
- In mathematical logic, meta-level definitions are expressed informally as definitions of mathematical functions.
- In program semantics, meta-level definitions are expressed in a convenient functional framework with a semantics that is easily defined and understood using informal mathematics. *Formal* denotational definitions are written in a mathematical meta-language corresponding to some formulation of a *Universal Domain* (a mathematical domain in which all relevant programming language domains can be simply embedded, usually as projections). This material is subject of a graduate level course on domain theory.
- A functional interpreter for language L written in a functional subset of L is called a *meta-circular* interpreter. It really isn't circular because it reduces the meaning of all programs to a single purely functional program which can be understood independently using simple mathematical machinery (inductive definitions over familiar mathematical domains).

Denotational Building Blocks

- Inductively defined ASTs for program syntax. We have thoroughly discussed this topic.
- What about denotations? For now, we will only use simple inductively defined values (without functional abstraction) like numbers, lists, tuples, etc.
- What about environments? Mathematicians like to use functions. An environment is a function from variables to denotations. But environment functions are special because they are *finite*. Software engineers prefer to represent them as lists of pairs, binding variables to denotations.
- In "higher-order" languages, functions are data objects. How do we represent them? For now we will use ASTs possibly supplemented by simple denotations (as described above).

Critique of Deferred Substitution Interpreter from Lecture 6

- How did we represent the denotations of **`lambda`**-abstractions (functions) in environments? By their ASTs. Is this implementation correct? No!
- Counterexample:

```
(let ([twice (lambda (f) (lambda (x) (f (f x))))])
  (let ([x 5])
    ((twice (lambda (y) (+ x y))) 0)))
```

Evaluate (syntactically)

```
(let [(twice (lambda (f) (lambda (x) (f (f x)))))]
  (let [(x 5)]
    ((twice (lambda (y) (+ x y))) 0)))

⇒ (let [(x 5)]
    (((lambda (f) (lambda (x) (f (f x))))
        (lambda (y) (+ x y)))
       0))

⇒ (((lambda (f) (lambda (x) (f (f x)))) (lambda (y) (+ 5 y)))
     0)

⇒ ((lambda (x) ((lambda (y) (+ 5 y)) ((lambda (y) (+ 5 y)) x))
     0)

⇒ ((lambda (y) (+ 5 y)) ((lambda (y) (+ 5 y)) 0))
⇒ ((lambda (y) (+ 5 y)) (+ 5 0))
⇒ ((lambda (y) (+ 5 y)) 5) ⇒ (+ 5 5) ⇒ 10
```

Evaluate (using our bad interpreter)

```
(let [(twice (lambda (f) (lambda (x) (f (f x)))))]
  (let (x 5)]
    (twice (lambda (y) (+ x y))) 0))  ⇒
{twice = (lambda (f) (lambda (x) (f (f x))))}
  (let [(x 5)] ((twice (lambda (y) (+ x y))) 0))  ⇒
{x = 5, twice = (lambda (f) (lambda (x) (f (f x))))}
  ((twice (lambda (y) (+ x y))) 0)  ⇒
{x = 5, ... }
  (((lambda (f) (lambda (x) (f (f x)))) (lambda (y) (+ x y))) 0) ⇒
{f = (lambda (y) (+ x y)),  x = 5, ... } ((lambda (x) (f (f x))) 0) ⇒
{x = 0, f = (lambda (y) (+ x y)), ... } (f (f x)) ⇒
{x = 0, f = (lambda (y) (+ x y)), ... } ((lambda (y) (+ x y)) (f x)) ⇒
{x = 0, ... } ((lambda (y) (+ x y)) ((lambda (y) (+ x y)) x)) ⇒
{x = 0, ... } ((lambda (y) (+ x y)) ((lambda (y) (+ x y)) 0)) ⇒
{y = 0, x = 0, ... } ((lambda (y) (+ x y)) (+ x y)) ⇒
{y = 0, x = 0, ... } ((lambda (y) (+ x y)) (+ 0 y)) ⇒
{y = 0, x = 0, ... } ((lambda (y) (+ x y)) (+ 0 0)) ⇒
{y = 0, x = 0, ... } ((lambda (y) (+ x y)) 0) ⇒
{y = 0, y = 0, x = 0, ... } (+ x y) ⇒ { y = 0, ... } (+ 0 y) ⇒
{ ... } (+ 0 0) ⇒ 0
```

Closures Are Essential!

- **Exercise**: evaluate the same expression using our broken interpreter. The computed "answer" is 0. The trace appears above!
- The interpreter uses the wrong binding for the free variable `x` in
  `(lambda (y) (+ x y))`
- The environment records deferred substitutions. When we pass a function as an argument, we need to pass a "package" including the deferred substitutions. Why? The function will be applied in a *different* environment which may associate the *wrong* bindings it free variables. In the PL (programming languages) literature, these packages (code representation + environment) are called *closures*.
- Note the similarity between this mistake and the "capture of bound variables". Unfortunately, this mistake has been labeled as a feature rather than a bug in much of the PL literature. It is called "dynamic scoping" rather than a horrendous mistake. Watch out whenever you must program in a language with "dynamic scoping".

Correct Semantic Interpreter

```
(define-struct (closure proc env)) ; closure is name of type
;; V = Const | Closure  ; revises our former definition of V
;; Binding = (make-Binding Sym V)  ; Note: Sym not Var
;; Env = (listOf Binding)          ; Lists are built-in to Scheme
;; Closure = (make-closure Proc Env)
;; R Env → V
(define eval
  (lambda (M env)
    (cond
      ((var? M) (lookup (var-name M) env))
      ((const? M) M)
      ((proc? M)) (make-closure M env))
      ((add? M)                           ; M has form (+ l r)
         (const-add (eval (add-left M) env) (eval (add-right M) env)))
      (else                               ; M has form (N1 N2)
         (apply (eval (app-rator M) env) (eval (app-rand M) env))))))

;; Closure V → V
(define apply
  (lambda (cl v)                          ; assume cl is a closure
    (eval (proc-body (closure-proc cl))
          (cons (make-binding (proc-param (closure-proc cl)) v)
                (closure-env cl)))
```

A Meta-Interpreter for CBN

- **Recall** the syntactic semantics for the CBN version of LC. What is different from our standard CBV (call-by-value) semantics for LC? What is our rule for reducing applications of program-defined functions (lambda-abstractions) `(lambda x M)`? Are there any restrictions on β-reduction?
- How do we implement CBN (unrestricted) β-reduction in a meta-interpreter. Recall that we must defer substitutions for parameters in the lambda-abstractions. How can we get the right answer even when we defer evaluation? What did we do in our CBV interpreter when we passed functions (lambda-abstractions) as argument values?
- What problem do closures eliminate? Finding the correct values for free variables in the bundled lambda-abstraction. In CBN we need to bind variables to unevaluated expressions, right? How can we avoid getting incorrect values for free variables in such expressions, just like we did for lambda-abstractions as values? You must answer this question to do Project 2.