Multi-sided rational surface patches with independent boundary control

Joe Warren

Department of Computer Science
Rice University
Houston, Tx 77584

Abstract

Various schemes have been proposed for creating rational surface patches with a variable number of sides. Although these schemes appear to be unrelated, they all produce surface parameterizations that have parameter values for which the numerator and denominator of the defining rational functions vanish simultaneously. These parameter values are *base points* of the rational parameterization.

This paper shows that any rational surface scheme permitting the independent control of non-adjacent boundary curves must produce parameterizations possessing base points. Locating these base points allows for a classification of possible multi-sided rational surface schemes.

1 Rational surfaces

A *rational* surface can be expressed in the form

$$\begin{aligned} x &= X(s,t), \\ y &= Y(s,t), \\ z &= Z(s,t), \\ w &= W(s,t) \end{aligned}$$

where X, Y, Z, and W are polynomials in s and t. The four-tuple (x, y, z, w) represents in homogeneous coordinates the affine point $(\frac{x}{w}, \frac{y}{w}, \frac{z}{w})$. Rational surfaces are popular in computer-aided geometric design (CAGD) for a variety of reasons. Rational surfaces are necessary to reproduce exactly quadric surfaces like the sphere or cylinder. Rational surfaces also afford greater shape control than do polynomial surfaces thanks to the presence of a non-constant denominator w.

Rational surfaces have also been used to solve several important problems in CAGD. For example, (Gregory 1974) uses rational surface patches to fill 4-sided holes smoothly even in the presence of incompatible edge twists. (Loop and DeRose 1989, Warren 1992) and others use rational surface patches to fill multi-sided holes.

Each of these techniques produces rational surfaces with an interesting property. Their defining parameterizations have at least one parameter value (u, v) for which

$$X(u, v) = Y(u, v) = Z(u, v) = W(u, v) = 0.$$

Such parameter values are *base points* of the defining parameterization.

The image of a base point is *not* necessarily a point at infinity. In fact, the image of a base point is a set of rational curves on the resulting rational surface (Clebsch 1868, Snyder et al. 1970). At a base point, the rational parameterization evaluates to $(\frac{0}{0}, \frac{0}{0}, \frac{0}{0})$. The image of a base point is a rational function of the direction in the parameter plane from which the base point is approached. This paper exposes the intrinsic connection between multi-sided rational surface patches and the theory of rational surfaces with base points.

2 Independent boundary control and rational surfaces

The need for multi-sided surface patches arises primarily because not all surface topologies can be represented using a tensor product mesh of rectangular surface patches. Various researchers (Charrot and Gregory 1984, Gregory 1983, Hosaka and Kimura 1984, Loop and DeRose 1989, Rockwood 1993, Sabin 1983, Warren 1992) have proposed methods for creating surface patches with more than four sides. Although these methods differ greatly, they share some important similarities. Each method produces a surface patch bounded by a set of curve segments. Typically, non-neighboring curve segments can be chosen independently. This property, known as *independent boundary control,* is necessary if the resulting patches are to be used to interpolate arbitrary networks of curves. For rectangular surface patches, independent boundary control implies that curves on opposite sides of the patch can be chosen independently. Tensor product Bézier surfaces are an example of a four-sided method that has this property.

The methods we consider are only those that produce a surface patch lying on a single rational surface (Charrot and Gregory 1984, Loop and DeRose 1989, Warren 1992). Methods like those of (Rockwood 1993) that produce a piecewise surface are not considered. This restriction to patches lying on a single rational surface allows us to apply some important techniques from algebraic geometry. For example, specifying a dense, bounded portion of a rational curve or surface determines the entire curve or surface (Fulton 1978, Walker 1950). No similar result holds for piecewise curves and surfaces.

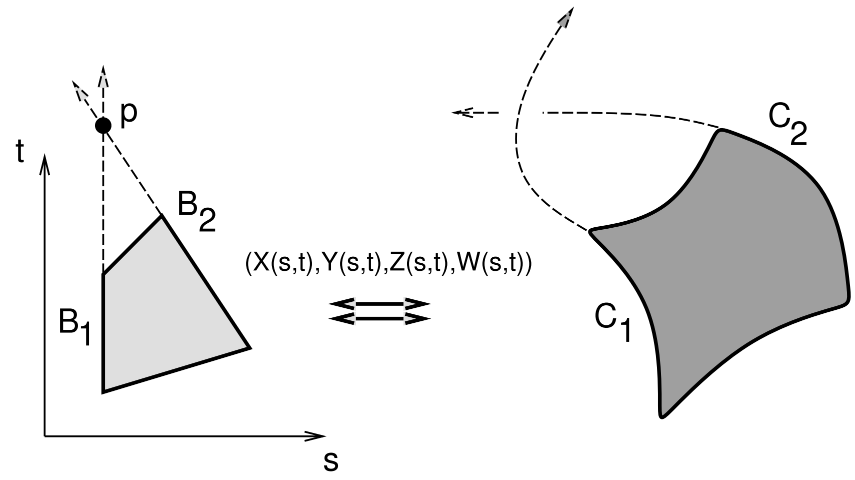

Figure 1. Two boundary curves and their inverses

The combination of independent boundary control and containment of the multi-sided patch in a single rational surface strongly restricts the types of surface patches possible. The following theorem, first suggested by (DeRose 1991), illustrates this limitation. The main consequence of this theorem is that only two types of multi-sided rational surface patches are possible. Subsequent sections illustrate these two types.

Theorem 1. *Let S be a rational surface interpolating two rational curves C_1 and C_2 that do not intersect in complex projective space. Then any rational parameterization σ for S must possess a base point.*

Proof. Let B_1 and B_2 be the preimages of C_1 and C_2, respectively, under σ. Since σ and C_1 are rational, B_1 must be the solution to a system of polynomial equations. In the case of bivariate parameterizations, B_1 must be either a point or an algebraic curve (Fulton 1978). If B_1 is a point, then σ must evaluate to $(0, 0, 0, 0)$ at B_1. Otherwise, the image B_1 under σ would be a point. Therefore, σ must have a base point at B_1. A similar argument holds for B_2.

If B_1 and B_2 are both algebraic curves, then by Bézout's theorem (Fulton 1978, Walker 1950), they must intersect in an least one point p in complex projective space. (See figure 1.) $\sigma(p)$ cannot be a single point since $\sigma(p)$ would necessarily be on both C_1 and C_2. This would contradict the fact that C_1 and C_2 are non-intersecting. Since σ is rational, $\sigma(p)$ must an algebraic curve. Therefore, p must be a base point of σ.

To illustrate the theorem, recall that tensor product Bézier surfaces possess independent boundary control. By theorem 1, these surfaces must have parameterizations that possess base points. Consider the following bilinear surface in s and t

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} x_{00} \\ y_{00} \\ z_{00} \\ 1 \end{pmatrix}(1-s)(1-t) + \begin{pmatrix} x_{01} \\ y_{01} \\ z_{01} \\ 1 \end{pmatrix}(1-s)(t) + \begin{pmatrix} x_{10} \\ y_{10} \\ z_{10} \\ 1 \end{pmatrix}(s)(1-t) + \begin{pmatrix} x_{11} \\ y_{11} \\ z_{11} \\ 1 \end{pmatrix}(s)(t).$$

The image of the lines $s = 0$ and $s = 1$ in parameter space is in general a pair of skew lines passing through the edges of the bilinear patch. By theorem 1, this parameterization must have a base point where $s = 0$ and $s = 1$ intersect. After homogenizing the parameterization (replacing 1's by u's), the resulting parameterization has a base point at $(s, t, u) = (0, 1, 0)$, the point at infinity where the parameter lines $s = 0$ and $s = 1$ intersect.

In fact, any multi-sided rational surface scheme with independent boundary control must produce rational parameterizations that have base points. The proof of theorem 1 allows for a classification of various multi-sided schemes. Base points arise in one of two ways in theorem 1. One possibility is that the image of a base point in the parametric domain is a bounding curve of the surface patch. The other possibility is that the preimages of two non-adjacent bounding curves intersect in a base point. Figure 2 illustrates these two cases.

The next two sections review two methods for creating multi-sided rational surface patches, each using one of these techniques. The first method creates a multi-sided patch as the image of a domain triangle by introducing base points at the corners of the parametric domain. The second method creates an n-sided patch as the image of an n-sided parametric domain. This construction introduces base points at the intersections of non-intersecting edges of the parametric domain.

3 Multi-sided surface patches: boundary base points

In this section, we describe a method for modeling four-, five-, and six-sided patches using a triangular parametric domain. We begin by briefly reviewing the Bernstein/Bézier method for representing bivariate rational functions.

Let P be a domain triangle with vertices v_0, v_1, and v_2. Barycentric coordinates for P consist of one basis function per vertex of P. These basis

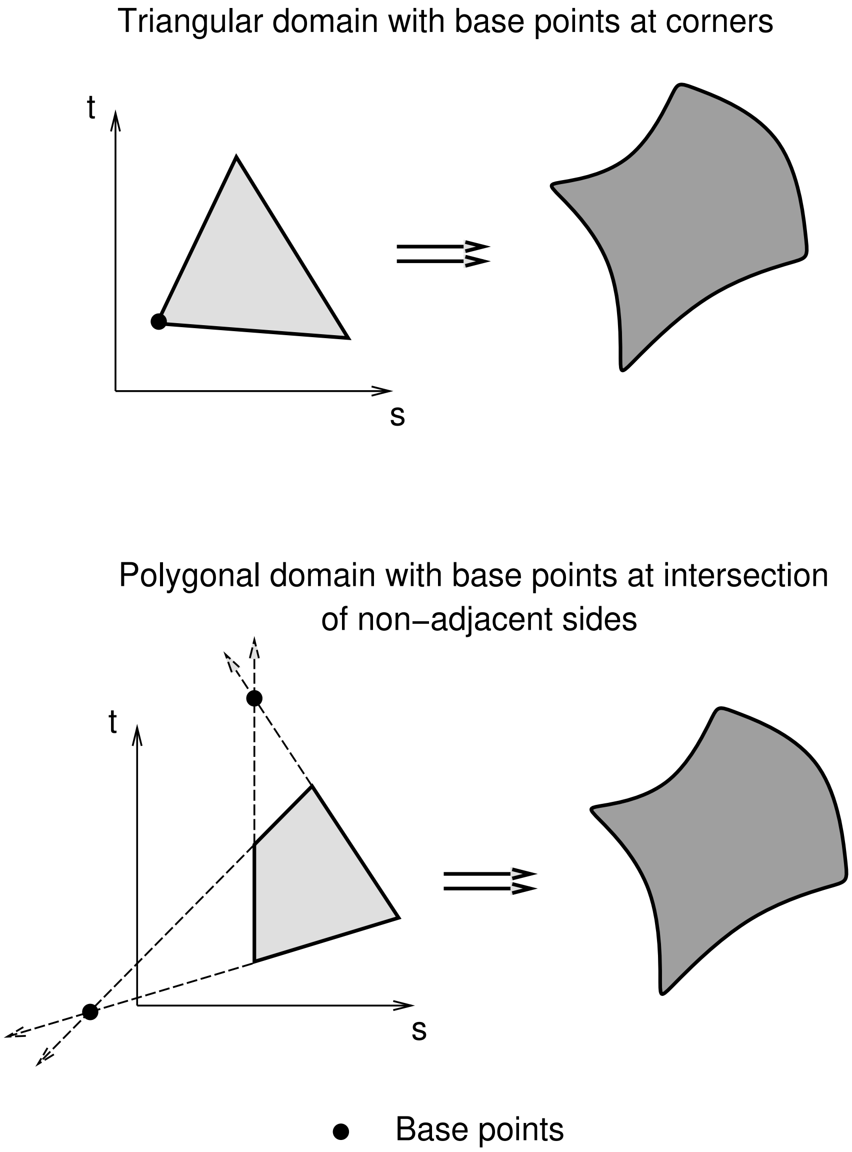

Figure 2. Two possible base point configurations

functions can be defined in terms of areas of subtriangles of P.

$$\beta_i(p) = \frac{area(\Delta\ v_{i-1}\ p\ v_{i+1})}{area(\Delta\ v_0\ v_1\ v_2)}.$$

(Here, the indices are taken modulo 3 and the area function is normalized to always return a positive value on the interior of P.) These barycentric coordinates $\beta_i(p)$ satisfy the following properties:

- Nonnegativity -
$$\beta_i(p) \geq 0 \quad \forall p \in P,$$
- Partition of unity -
$$\sum_{i=0}^{2} \beta_i(p) = 1 \quad \forall p \in P,$$
- Linear precision -
$$\sum_{i=0}^{2} v_i \beta_i(p) = p \quad \forall p \in P.$$

The Bernstein/Bézier basis functions of degree k are simply terms of the multinomial expansion of

$$(\beta_0(p) + \beta_1(p) + \beta_2(p))^k.$$

These basis functions are explicitly expressed as

$$B_{i_0 i_1 i_2}(p) = \frac{k!}{i_0! i_1! i_2!} \beta_0^{i_0} \beta_1^{i_1} \beta_2^{i_2}$$

where i_0, i_1, and i_2 are nonnegative and sum to k. The functions $B_{i_0 i_1 i_2}(p)$ are nonnegative and form a partition of unity due to the corresponding property for barycentric coordinates. Likewise, the linear precision property for barycentric coordinates implies that the Bernstein/Bézier functions of degree k form a basis for the polynomials of degree k (Loop and DeRose 1989).

A *rational* Bézier surface of degree k has a parameterization of the form:

$$\begin{pmatrix} \mathbf{p} \\ w \end{pmatrix} = \sum_{\substack{i_0, i_1, i_2 \geq 0 \\ i_0 + i_1 + i_2 = k}} \begin{pmatrix} w_{i_0 i_1 i_2} \mathbf{p}_{i_0 i_1 i_2} \\ w_{i_0 i_1 i_2} \end{pmatrix} B_{i_0 i_1 i_2}(p). \tag{3.1}$$

A rational Bézier surface patch is the image of P under this parameterization. Points on the surface patch are represented in homogeneous coordinates $(\mathbf{p}, w) = (x, y, z, w)$. The points

$$\mathbf{p}_{i_0 i_1 i_2} = (x_{i_0 i_1 i_2}, y_{i_0 i_1 i_2}, z_{i_0 i_1 i_2})$$

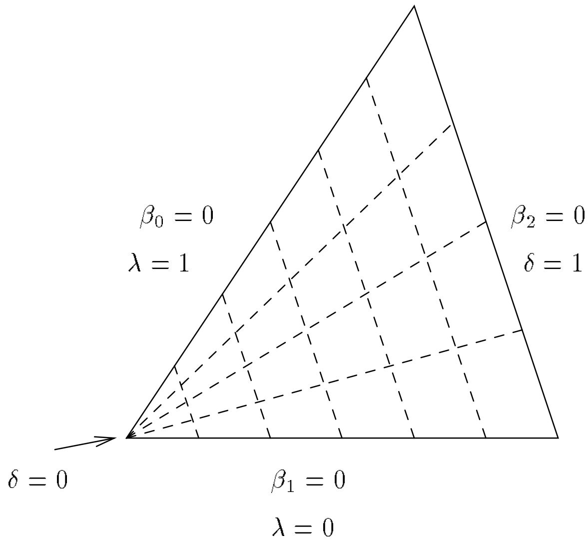

Figure 3. Transforming barycentric coordinates to Cartesian coordinates

are control points, and associated with each is a *weight* $w_{i_0 i_1 i_2}$. The set of all control points and their weights forms a weighted control polyhedron.

(Warren 1992) has shown that introducing a base point into this parameterization at vertex v_2 of P results in a surface patch with four, not three, sides. The effect of this base point on the surface patch can be understood by reparameterizing the triangular patch as a degenerate rectangular surface using the following transformation.

$$\begin{aligned} \lambda &= \tfrac{\beta_1}{\beta_0+\beta_1}, \\ \delta &= 1 - \beta_2. \end{aligned} \tag{3.2}$$

This transformation maps barycentric coordinates $(\beta_0, \beta_1, \beta_2)$ to Cartesian coordinates (λ, δ). Figure 3 shows this transformation. The dashed lines are the new Cartesian coordinates mapped onto P. Mathematically, this transformation blows up the vertex v_2 with barycentric coordinates $(0, 0, 1)$ into the edge $\delta = 0$ in Cartesian coordinates. The inverse transformation is

$$\begin{aligned} \beta_0 &= (1-\lambda)\delta, \\ \beta_1 &= \lambda\delta, \\ \beta_2 &= 1-\delta. \end{aligned} \tag{3.3}$$

Substituting the transformation of equation 3.3 into the definition of a rational triangular surface in equation 3.1 yields a tensor product surface defined over the unit square $0 \leq \lambda, \delta \leq 1$. In the absence of a base point at v_2, the side of the rectangular patch corresponding to $\delta = 0$ degenerates into the point $\mathbf{p}_{00k}$ and the patch retains three sides. In the presence of a base point at v_2, the image of $\delta = 0$ is a curve and the resulting patch has four sides.

The introduction of base points is particularly easy in the rational Bézier form. Specifically, the parameterization of equation 3.1 has a base point at v_2 if and only if the weight w_{00k} is zero. Understanding the effect of this base point on the surface patch is easy in the rational Bézier form. For example, consider a triangular rational quadratic Bézier surface:

$$\begin{aligned}\begin{pmatrix}\mathbf{p}\\ w\end{pmatrix} &= \begin{pmatrix}w_{002}\mathbf{p}_{002}\\ w_{002}\end{pmatrix}\gamma^2 + \begin{pmatrix}w_{101}\mathbf{p}_{101}\\ w_{101}\end{pmatrix}2\alpha\gamma + \begin{pmatrix}w_{011}\mathbf{p}_{011}\\ w_{011}\end{pmatrix}2\beta\gamma + \\ &\quad \begin{pmatrix}w_{200}\mathbf{p}_{200}\\ w_{200}\end{pmatrix}\alpha^2 + \begin{pmatrix}w_{110}\mathbf{p}_{110}\\ w_{110}\end{pmatrix}2\alpha\beta + \begin{pmatrix}w_{020}\mathbf{p}_{020}\\ w_{020}\end{pmatrix}\beta^2.\end{aligned}$$

Setting w_{002} to zero and applying transformation 3.3 to this triangular surface yields the following tensor product rational quadratic surface:

$$\begin{pmatrix}\mathbf{p}\\ w\end{pmatrix} = (\begin{pmatrix}w_{101}\mathbf{p}_{101}\\ w_{101}\end{pmatrix}(1-\lambda) + \begin{pmatrix}w_{011}\mathbf{p}_{011}\\ w_{011}\end{pmatrix}\lambda)(2(1-\delta)) +$$
$$(\begin{pmatrix}w_{200}\mathbf{p}_{200}\\ w_{200}\end{pmatrix}(1-\lambda)^2 + \begin{pmatrix}w_{110}\mathbf{p}_{110}\\ w_{110}\end{pmatrix}2(1-\lambda)\lambda + \begin{pmatrix}w_{020}\mathbf{p}_{020}\\ w_{020}\end{pmatrix}\lambda^2)\delta.$$

Note that a common factor of δ has been canceled out of the parameterization. The image of the base point at v_2 is the edge between $\mathbf{p}_{101}$ and $\mathbf{p}_{011}$. Setting the weight w_{002} to zero has the effect of "clipping" away the control point $\mathbf{p}_{002}$ from the control polyhedron.

More generally, if all weights $w_{i_0 i_1 i_2}$ for which $i_0 + i_1 < m$ are set to zero where $m < k-1$, the image of the base point at v_2 is a rational Bézier curve of degree m with control points $\mathbf{p}_{i_0 i_1 i_2}$ where $i_0 + i_1 = m$. This technique can also create five- and six-sided patches if applied to each vertex of the triangular domain independently. The clipping paradigm described above provides a simple way of understanding the boundary behavior of these patches. If S is a rational Bézier surface with a control polyhedron Q, then deleting those control points whose weights have been set to zero as described above yields a new control polyhedron $\hat{Q}$. The boundary curves of S are rational Bézier curves defined by those control points along the corresponding boundaries of $\hat{Q}$. S and $\hat{Q}$ still retain all the standard properties of rational Bézier surfaces, such as the convex hull property, when all of the $w_{i_0 i_1 i_2}$ are nonnegative.

Such a five- or six-sided patch can be converted into an equivalent set of patches free of base points by subdividing the patch into four patches and applying transformation 3.3 to each corner patch independently. Figure 4 shows an example of this process in the case of a quartic patch with

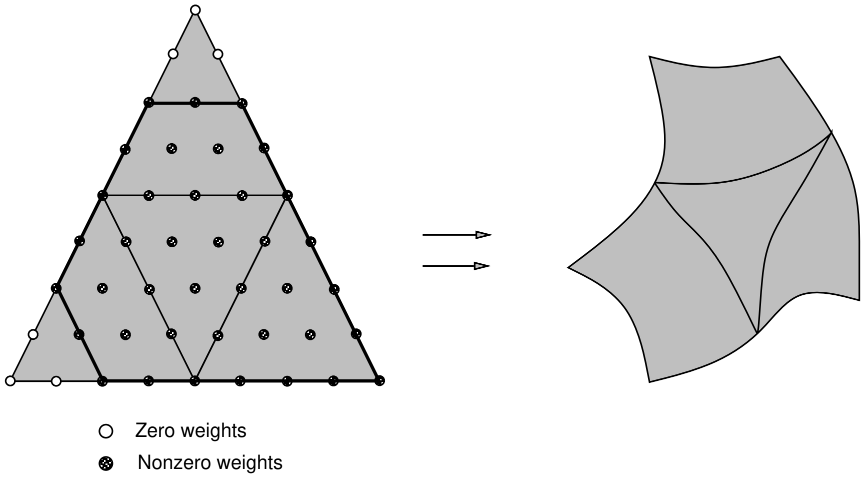

Figure 4. Transforming a five-sided patch into a collection of 3- and 4-sided patches

base points at two of its vertices. The result is a collection of three- and four-sided patches.

4 Multi-sided surface patches: external base points

As seen in the previous section, a multi-sided rational surface can be created as the image of a triangle in the parametric domain. An alternative approach is to build an n-sided rational surface patch that is the image of an n-sided parametric domain. (Charrot and Gregory 1984) takes this approach in creating pentagonal surface patches. (Loop and DeRose 1989) generalizes this approach to arbitrary polygonal domains. (Wachspress 1975) gives a similar construction in terms of finite elements.

The key to the construction of Loop and DeRose is the generalization of barycentric coordinates to arbitrary convex polygons. These coordinate functions should be nonnegative on the polygonal domain, form a partition of unity, and have linear precision. Given these properties, higher order basis functions can be constructed by application of the multinomial theorem, as in the triangular case.

Let P be a convex n-gon with vertices $v_0, v_1, \ldots, v_{n-1}$. Barycentric coordinates over P are rational functions of the form:

$$\beta_i(p) = \frac{b_i(p)}{d(p)}$$

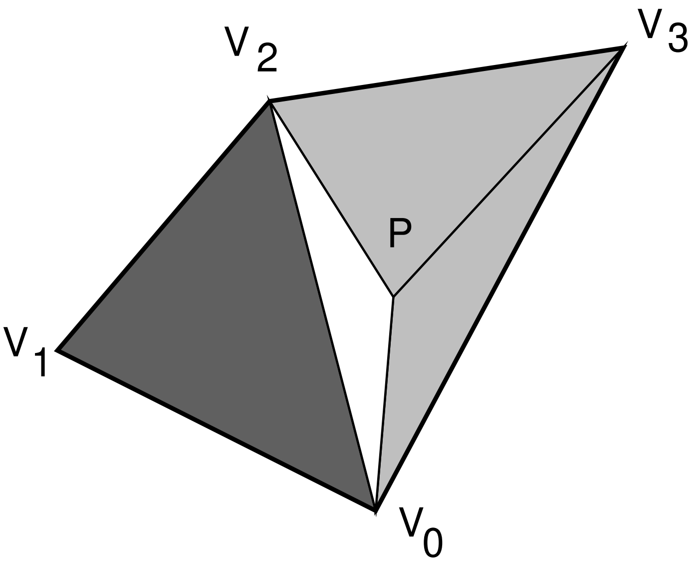

Figure 5. Barycentric coordinates for a quadrilateral

where

$$\begin{aligned} b_i(p) &= area(\Delta\ v_{i-1}\ v_i\ v_{i+1}) \prod_{j=i+1}^{i+n-2} area(\Delta\ v_j\ p\ v_{j+1}) \\ d(p) &= \sum_{i=0}^{n-1} b_i(p). \end{aligned}$$

(Note that all indices are taken modulo n.)

The $\beta_i(p)$ are nonnegative and form a partition of unity as direct consequence of their definition. Loop and DeRose give a proof of linear precision in the case when P is regular. Warren and DeRose have proven the result for arbitrary convex polygons P (Warren and DeRose 1993).

Figure 5 illustrates the numerator of the barycentric coordinate function for the vertex v_1 of a quadrilateral. The area of the dark triangle is the leading constant in $b_1(p)$. The areas of the two light triangles form the terms of the product in $b_1(p)$. If P is the unit rectangle, then the sum of the $b_i(p)$ is one and the $\beta_i(p)$ degenerate into the standard bilinear basis functions.

As in the triangular case, basis functions of order k are terms of the multinomial expansion of

$$(\beta_0(p) + \beta_1(p) + \ldots + \beta_{n-1}(p))^k.$$

(Here, the multinomial expansion is equivalent to composition with a high-dimensional Bézier simplex as demonstrated by Loop and DeRose.)

These basis functions can be explicitly expressed as

$$B_{i_0 i_1 \ldots i_{n-1}}(p) = \frac{k!}{i_0! i_1! \ldots i_{n-1}!} \beta_0^{i_0} \beta_1^{i_1} \ldots \beta_{n-1}^{i_{n-1}}$$

where i_0, i_1, ..., i_{n-1} are nonnegative and sum to k.

The basis functions $B_{i_0 i_1 \ldots i_{n-1}}(p)$ are nonnegative over P and form a partition of unity. Again, linear precision for the $\beta_i(p)$ is sufficient to guarantee that the order k basis functions span the space of all polynomials of degree k.

An *S-patch* of order k has the form:

$$\mathbf{p} = \sum_{\substack{i_0, i_1, \ldots, i_{n-1} \geq 0 \\ i_0 + i_1 + \ldots + i_{n-1} = k}} \mathbf{p}_{i_0 i_1 \ldots i_{n-1}} B_{i_0 i_1 \ldots i_{n-1}}(p). \tag{4.1}$$

The points $\mathbf{p}_{i_0 i_1 \ldots i_{n-1}} = (x_{i_0 i_1 \ldots i_{n-1}}, y_{i_0 i_1 \ldots i_{n-1}}, z_{i_0 i_1 \ldots i_{n-1}})$ are control points of the S-patch. The set of all control points forms the control polyhedron associated with the S-patch.

As in the case of triangular Bézier surfaces, S-patches are closely related to their control polyhedra. The control polyhedron approximates the S-patch. The S-patch lies in the convex hull of its control polyhedron. The boundary behavior of S-patches is particularly simple. The image of the edge of P from vertex v_j to v_{j+1} is a Bézier curve with control points $\mathbf{p}_{0 \ldots 0 i_j i_{j+1} 0 \ldots 0}$ where $i_j + i_{j+1} = k$.

Since the non-adjacent boundary curves of an S-patch are independent, the parameterization for an S-patch must have base points. These base points are at the intersections of non-adjacent edges in the parametric domain. Figure 6 illustrates this situation in the case of a pentagonal patch. The five edges of the parametric domain P are extended to form a five-pointed star. Each of the barycentric coordinate functions $\beta_i(p)$ for P has base points at the vertices of the star.

Recall that the numerators of the $\beta_i(p)$ were of the form

$$b_i(p) = area(\Delta\ v_{i-1}\ v_i\ v_{i+1} \prod_{j=i+1}^{i+n-2} area(\Delta\ v_j\ p\ v_{j+1}).$$

If p lies at the intersection of non-adjacent lines $v_c v_{c+1}$ and $v_d v_{d+1}$, then the areas of the triangles $v_c p v_{c+1}$ and $v_d p v_{d+1}$ both vanish. Since one of these areas appears in the product defining $b_i(p)$ for all i, $b_i(p)$ must vanish at p for all i. Since $d(p)$, the denominator of $\beta_i(p)$, is simply the sum of the $b_i(p)$, it too must vanish. Therefore, for all i, $\beta_i(p)$ must have a base point at the intersection of non-adjacent edges of P.

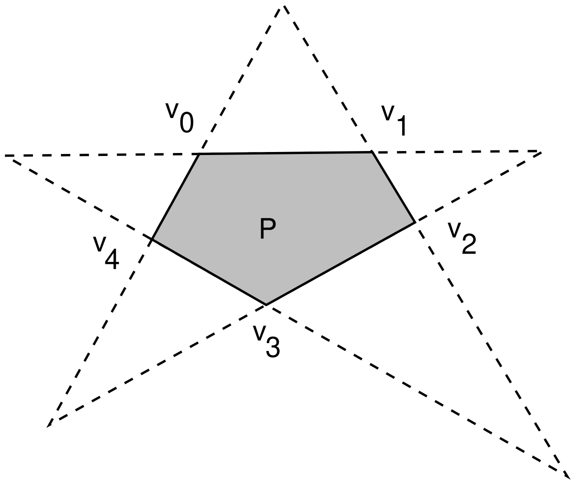

Figure 6. Base points for a pentagonal S-patch

5 Conclusions and future work

This paper has explored the theoretical underpinnings of several techniques for creating multi-sided rational surface patches. Specifically, it has illuminated the relationship between independent boundary control and base points. The presentation of two example methods that use rational parameterizations with base points has illustrated this relationship.

Warren's method relies on introducing base points into rational parameterizations over triangular domains. This technique has the advantage that four-, five-, and six-sided patches can be defined using only standard triangular basis functions. A similar method can be used in conjunction with tensor product parameterizations over rectangular domains to create patches with up to eight sides. However, building patches with more than eight sides is much more difficult.

S-patches are completely general in this regard. An n-sided patch is simply the image of an n-sided domain. However, this generality is at the expense of using barycentric coordinates over polygonal domains. These basis functions are intrinsically rational and represent an extra level of complexity over the triangular case.

One interesting avenue of exploration would be combining these two techniques. Extending S-patches to the rational case is straightforward. Zeroing the weight of a corner control point of an S-patch has the same effect as in the triangular case, the introduction of an extra edge into the patch. One possible application of this idea is in constructing multi-sided incompatible edge twist patches. Certain incompatible edge twist patches such as Gregory's patch (Gregory 1974) incorporate base points at the corners of their parametric domains. A triangular Gregory patch can be viewed as a six-sided patch built using Warren's method whose newly created edges are collapsed back into degenerate vertices. (See (Warren 1992) for more details). A similar construction could be attempted using S-patches in place of triangular Bézier surfaces.

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