An Extension of Chaiken’s Algorithm to B-Spline Curves with Knots in Geometric Progression

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Abstract

Chaiken’s algorithm is a procedure for inserting new knots into uniform quadratic B-spline curves by doubling the control points and taking two successive averages. Lane and Riesenfeld showed that Chaiken’s algorithm extends to uniform B-spline curves of arbitrary degree. By generalizing the notion of successive averaging, we further extend Chaiken’s algorithm to B-spline curves of arbitrary degree for knot sequences in geometric and affine progression.

1 Subdivision for knots in arithmetic progression

Let $N^{n+1}_k(t)$ be the B-spline basis function of degree $n+1$ whose support lies over the knot sequence $t_{2k}, t_{2k+2}, \ldots, t_{2k+2n+4}$ and let $\tilde{N}^{n+1}_k(t)$ be the B-spline basis function of degree $n+1$ whose support lies over the refined knot sequence $t_k, t_{k+1}, \ldots, t_{k+n+2}$. Since the B-splines form a basis, there exist constants $\alpha_{j+1}^{n}$ such that

$$N^{n+1}_k(t) = \sum_{j=0}^{n+2} \alpha_j^{n+1} \tilde{N}^{n+1}_{j+2k}(t).$$

(1)

For the degree zero basis functions, the $\alpha$’s satisfy

$$\alpha_0^0 = \alpha_1^0 = 1,$$

(2)

but for arbitrary knot sequences, the $\alpha$’s depend on $k$.

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In the case of uniform (arithmetic) knot sequences, Lane and Riesenfeld [LR80, Rie75] observed that the $\alpha_j^{n+1}$s are independent of $k$. For knot sequences satisfying $t_{i+1} = t_i + \gamma$, the B-spline basis functions satisfy the identities

$$N_0^{n+1}(t - 2k\gamma) = N_k^{n+1}(t),$$
$$\hat{N}_0^{n+1}(t - k\gamma) = \hat{N}_k^{n+1}(t).$$

If equation 1 holds for $k = 0$, then for any $k$

$$N_k^{n+1}(t) = N_0^{n+1}(t - 2k\gamma),$$
$$= \sum_{j=0}^{n+2} \alpha_j^{n+1} \hat{N}_j^{n+1}(t - 2k\gamma)$$
$$= \sum_{j=0}^{n+2} \alpha_j^{n+1} \hat{N}_{j+2k}^{n+1}(t).$$

Therefore in the uniform case, any formula for subdividing $N_0^n$ is automatically a formula for subdividing $N_k^n$. Lane and Riesenfeld [LR80] then observed that the following recurrence holds among the $\alpha$’s.

**Theorem 1** For any knot sequence satisfying $t_{i+1} = t_i + \gamma$, the $\alpha$’s satisfy the recurrence

$$\alpha_j^{n+1} = \frac{1}{2} \alpha_{j-1}^n + \frac{1}{2} \alpha_j^n. \quad (3)$$

This recurrence leads directly to a subdivision (knot insertion) algorithm for B-spline curves with knots in arithmetic progression. To illustrate this algorithm, consider the case of a quadratic B-spline curve with a single nonzero control point $P_k$

$$S(t) = P_k \cdot N_k^2(t).$$

Recurrence 3 can be used to compute the new non-zero control points $Q_k, Q_{k+1}, Q_{k+2}$, and $Q_{k+3}$ of the subdivided B-spline curve satisfying

$$S(t) = Q_k \hat{N}_k^2(t) + Q_{k+1} \hat{N}_{k+1}^2(t) + Q_{k+2} \hat{N}_{k+2}^2(t) + Q_{k+3} \hat{N}_{k+3}^2(t).$$

Figure 1 illustrates the diagrams for three separate recurrences starting at $P_0, P_1,$ and $P_2$. In the degree zero case, the $\alpha^0$’s satisfy $\alpha_0^0 = \alpha_1^0 = 1$. Therefore, the topmost level of the diagrams contain two copies of $P_k$ By equation 3, the $\alpha^{n+1}$’s can be computed from $\alpha^n$’s via averaging. Therefore, control points at $(n + 1)$st level of the diagrams can be computed
from control points on the \( n \)th level of the diagrams via averaging. Edges in the diagrams are labeled by an associated multiplier \( \frac{1}{2} \). The control points for the subdivided B-spline curve are produced on the bottom level of the diagram.

Given a B-spline curve with control points \( P_k \)

\[
S(t) = \sum_k P_k N_k^{n+1}(t),
\]

the Lane-Riesenfeld algorithm computes new control points \( Q_k \) over a refined knot sequence that defines the same curve

\[
S(t) = \sum_k Q_k \hat{N}_k^{n+1}(t).
\]

In general, several adjacent control points in the original B-spline curve may contribute to the same control point in the subdivided B-spline curve. To combine these contributions, the recurrence for each individual basis function can be overlapped to form a single global recurrence for all the basis functions (see figure 2). In the quadratic case, the resulting method is exactly Chaiken’s algorithm [Cha74, Rie75].

This paper investigates other types of knot sequences for which the corresponding B-spline curves have analogous subdivision algorithms. In particular, we show that B-spline curves defined over knot sequences in geometric progression possess a similar subdivision
algorithm. In the process of proving this algorithm, a new proof of the Lane-Riesenfeld algorithm is derived.

2 Subdivision for knots in geometric progression

A knot sequence is in geometric progression if \( t_{i+1} = \beta t_i \) where \( \beta \neq 1 \). For geometric knot sequences, the basis functions are related by the identities

\[
N_0^{n+1}(t/\beta^{2k}) = N_k^{n+1}(t),
\]

\[
\hat{N}_0^{n+1}(t/\beta^k) = \hat{N}_k^{n+1}(t).
\]

If equation 1 holds for \( k = 0 \), then for any \( k \)

\[
N_k^{n+1}(t) = N_0^{n+1}(t/\beta^{2k}),
\]

\[
= \sum_{j=0}^{n+2} a_j^{n+1} \hat{N}_j^{n+1}(t/\beta^{2k})
\]

\[
= \sum_{j=0}^{n+2} a_j^{n+1} \hat{N}_{j+2k}^{n+1}(t).
\]

Therefore, any subdivision formula for \( N_0^{n+1} \) is automatically a subdivision formula for \( N_k^{n+1} \).

In the case of knots in geometric progression, the \( \alpha \)'s of equation 1 can be computed using the following theorem.

**Theorem 2** For knot sequences satisfying \( t_{i+1} = \beta t_i \), the \( \alpha \)'s satisfy the recurrence

\[
\alpha_j^{n+1} = \frac{\beta^{n+1}}{1 + \beta^{n+1}} a_j^{n} + \frac{1}{1 + \beta^{n+1}} a_j^{n}.
\]

\[ (4) \]

**Figure 3:** Subdivision recurrences for single basis functions
A subdivision algorithm similar to the Lane-Riesenfeld algorithm can now be derived in manner similar to that of the previous section. Applying Theorem 2 to a single basis function \( P_k N_k^n(t) \) yields the recurrences shown in figure 3. Note that the multipliers on the \( k \)th level of the diagram are \( \frac{\beta_k^k}{1+\beta_k} \) and \( \frac{1}{1+\beta_k} \). Contributions of neighboring basis functions from the original B-spline curve to the same basis function in the subdivided B-spline curve can be computed by overlapping the recurrences as shown in figure 4.

More generally, if the B-spline curve is of the form

\[
S(t) = \sum_k P_k N_k^n+1(t),
\]

then the subdivision algorithm proceeds as follows:

1. Construct a new sequence of control points \( Q_{2k}^0 = Q_{2k+1}^0 = P_k \).

2. For \( j = 1 \) to \( n+1 \),
   
   (a) For all control points \( Q_k^j \), let
   
   \[
   Q_k^j = \frac{\beta_j}{1+\beta_j} Q_k^{j-1} + \frac{1}{1+\beta_j} Q_{k+1}^{j-1}.
   \]

   The resulting control points now satisfy

\[
S(t) = \sum_k Q_k^{n+1} N_k^n(t).
\]

3 **A proof using Prautsch’s method**

To prove Theorems 1 and 2, we use a variant of Prautsch’s proof of the Oslo algorithm [Pra84]. The key to this proof is to apply the subdivision recurrence of equation 1 and the Cox-de Boor degree recurrence to \( N_k^n(t) \). Two different expressions result, depending on

![Figure 4: A subdivision method for knots in geometric progression](image-url)
the order in which the two equations are applied to \( N_k^{n+1}(t) \). The necessary degree recurrence is found in [dB72].

\[
N_k^{n+1}(t) = \frac{t - t_{2k}}{t_{2k+2n+2} - t_{2k}} N_k^n(t) + \frac{t_{2k+2n+4} - t}{t_{2k+2n+4} - t_{2k+2}} N_{k+1}^n(t).
\]  

(5)

The proof will proceed over an arbitrary knot sequence with specialization to particular knot sequences deferred till appropriate.

First, we apply the subdivision formula of equation 1 to \( N_k^n \) and \( N_{k+1}^n \) in equation 5.

\[
N_k^{n+1}(t) = \sum_{j=0}^{n+2} \frac{t - t_{2k}}{t_{2k+2n+2} - t_{2k}} \alpha_j^n + \frac{t_{2k+2n+4} - t}{t_{2k+2n+4} - t_{2k+2}} \alpha_j^{n-2} \hat{N}_{j+2k}^n(t).
\]  

(6)

Next, we apply the degree recurrence of equation 5 to the basis function \( \hat{N}_j^n \) of equation 1.

\[
N_k^{n+1}(t) = \sum_{j=0}^{n+2} \frac{t - t_{2k+j}}{t_{2k+j+n+1} - t_{2k+j}} \alpha_j^{n+1} + \frac{t_{2k+j+n+1} - t}{t_{2k+j+n+1} - t_{2k+j}} \alpha_j^{n-1} \hat{N}_{j+2k}^n(t).
\]  

(7)

Comparing the coefficients of \( \hat{N}_j^n(t) \) in equations 6 and 7 yields the following relation among the \( \alpha \)'s.

\[
\frac{t - t_{2k}}{t_{2k+2n+2} - t_{2k}} \alpha_j^n + \frac{t_{2k+2n+4} - t}{t_{2k+2n+4} - t_{2k+2}} \alpha_j^{n-2} = \frac{t - t_{2k+j}}{t_{2k+j+n+1} - t_{2k+j}} \alpha_j^{n+1} + \frac{t_{2k+j+n+1} - t}{t_{2k+j+n+1} - t_{2k+j}} \alpha_j^{n-1}.
\]

(8)

This is the fundamental recurrence from which proofs of both theorems will be derived.

### 3.1 Knots in arithmetic progression

The Lane-Riesenfeld algorithm works for knots in arithmetic progression, \( t_{i+1} = t_i + \gamma \). If we assume for the sake of simplicity that \( t_0 = 0 \) and \( \gamma = 1 \), then sequence satisfies \( t_i = i \).

Equation 8 simplifies to

\[
\frac{t - 2k}{2n + 2} \alpha_j^n + \frac{2k + 2n + 4 - t}{2n + 2} \alpha_j^{n-2} = \frac{t - (2k + j)}{n + 1} \alpha_j^{n+1} + \frac{2k + j + n + 1 - t}{n + 1} \alpha_j^{n-1}.
\]

Since this equality holds for all \( t \), the \( t \) coefficients must be equal.

\[
\frac{\alpha_j^n}{2} - \frac{\alpha_j^{n-2}}{2} = \alpha_j^{n+1} - \alpha_j^{n-1}.
\]  

(9)

Note that this recurrence is independent of \( k \).

Equation 3 can be derived from equation 9 by induction on \( j \). For \( j = 0 \), equation 9 simplifies to

\[
\alpha_0^{n+1} = \frac{\alpha_0^n}{2}.
\]
If equation 3 holds for \( j - 1 \), then

\[
\alpha_{j-1}^{n+1} = \frac{\alpha_{j-2}^n}{2} + \frac{\alpha_{j-1}^n}{2}.
\]

Substituting this expression into equation 9 yields

\[
\alpha_j^{n+1} = \frac{\alpha_j^n}{2} - \frac{\alpha_{j-2}^n}{2} + \left( \frac{\alpha_{j-2}^n}{2} + \frac{\alpha_{j-1}^n}{2} \right) = \frac{\alpha_j^n}{2} + \frac{\alpha_{j-1}^n}{2}.
\]

Therefore, equation 3 holds for all \( j \).

### 3.2 Knots in geometric progression

Equation 8 is also crucial in proving Theorem 2 correct. If the knots are in geometric progression, \( t_{i+1} = \beta t_i \); setting \( t_0 = 1 \) yields that \( t_i = \beta^i \). After simplification, the constant terms of equation 8 satisfy the relation:

\[
- \frac{1}{1 + \beta^{n+1}} \alpha_j^n + \frac{\beta^{2n+2}}{1 + \beta^{n+1}} \alpha_j^{n-2} = -\alpha_j^{n+1} + \beta^{n+1} \alpha_j^{n-1}.
\]  

(10)

Again, equation 4 can be derived from equation 10 by induction on \( j \). For \( j = 0 \), the formulas trivially agree. If equation 4 holds for \( j - 1 \), then

\[
\alpha_{j-1}^{n+1} = \frac{\beta^{n+1}}{1 + \beta^{n+1}} \alpha_{j-2}^n + \frac{1}{1 + \beta^{n+1}} \alpha_{j-1}^n.
\]

Substituting this expression into equation 10 yields

\[
\alpha_j^{n+1} = \beta^{n+1} \left( \frac{\beta^{n+1}}{1 + \beta^{n+1}} \alpha_{j-2}^n + \frac{1}{1 + \beta^{n+1}} \alpha_{j-1}^n \right) + \frac{1}{1 + \beta^{n+1}} \alpha_j^n - \frac{\beta^{2n+2}}{1 + \beta^{n+1}} \alpha_{j-2}^n,
\]

\[
= \frac{\beta^{n+1}}{1 + \beta^{n+1}} \alpha_{j-1}^n + \frac{1}{1 + \beta^{n+1}} \alpha_j^n.
\]

Therefore, equation 4 holds for all \( j \).

Applying a similar approach to the \( t \) terms of equation 8 yields a second recurrence among the \( \alpha \)'s.

\[
\alpha_j^{n+1} = \frac{\beta^j}{1 + \beta^{n+1}} \alpha_{j-1}^n + \frac{\beta^{j-1}}{1 + \beta^{n+1}} \alpha_{j-1}^n.
\]

Combining this equation and equation 4 yields a recurrence relating \( \alpha_j^n \) to \( \alpha_{j-1}^n \):

\[
\alpha_j^n = \frac{\beta^{j-1}(1 - \beta^{n-j+2})}{1 - \beta^j} \alpha_{j-1}^n.
\]

This formula in conjunction with the fact that

\[
\alpha_0^n = \prod_{k=1}^{n} \frac{1}{1 + \beta^k}
\]

allows for fast explicit construction of the \( \alpha \)'s.
4 Subdivision for knots in affine progression

Arithmetic and geometric progressions may be viewed as special instances of a more general type of sequence, the affine progression:

\[ t_{i+1} = at_i + \delta \]

A simple example of an affine progression is the sequence 0, $\frac{1}{2}$, $\frac{3}{4}$, $\frac{7}{8}$, $\frac{15}{16}$, ... . This sequence satisfies the relation $t_{i+1} = \frac{1}{2}t_i + \frac{1}{2}$.

If the original knot sequence $t_0, t_2, t_4, t_6, ...$ satisfies the relation

\[ t_{i+2} = \beta^2 t_i + 2\gamma, \]

then for $\beta = 1$ the resulting $t_i$ lie in arithmetic progression. For $\gamma = 0$, the $t_i$ lie in geometric progression. This knot sequence can be refined to create a new knot sequence $t_0, t_1, t_2, t_3, ...$ in affine progression via the following relation:

\[ t_{i+1} = \beta t_i + \frac{2\gamma}{1 + \beta} \] (11)

Note that $t_0, t_2, t_4, ...$ still lie in the original affine progression.

\[
\begin{align*}
t_{i+2} &= \beta t_{i+1} + \frac{2\gamma}{1 + \beta}, \\
&= \beta(\beta t_i + \frac{2\gamma}{1 + \beta}) + \frac{2\gamma}{1 + \beta}, \\
&= \beta^2 t_i + 2\gamma(\frac{\beta}{1 + \beta} + \frac{1}{1 + \beta}), \\
&= \beta^2 t_i + 2\gamma.
\end{align*}
\]

When $\beta \neq 1$, affine progressions and geometric progressions are closely related. Specifically, the elements of an affine progression can be translated to bring the sequence into a geometric progression. Given the affine progression of equation 11, we define a translate of this sequence as follows:

\[ \hat{t}_i = t_i + \frac{2\gamma}{\beta^2 - 1}, \] (12)

This new sequence of $\hat{t}_i$'s lies in geometric progression since

\[
\begin{align*}
\hat{t}_{i+1} &= t_{i+1} + \frac{2\gamma}{\beta^2 - 1}, \\
&= \beta t_i + \frac{2\gamma}{1 + \beta} + \frac{2\gamma}{\beta^2 - 1},
\end{align*}
\]
\[
\beta(t_i + \frac{2\gamma}{\beta^2 - 1}),
\]
\[
= \beta t_i.
\]

Because of this observation, the subdivision algorithm of section 2 can be applied to B-spline curves whose knots are in affine progression. For \( \beta = 1 \), the resulting knots are in arithmetic progression. Substituting \( \beta = 1 \) into the recurrence 4 yields the Lane-Riesenfeld recurrence of equation 3. Thus, the subdivision algorithm of section 2 degenerates into the Lane-Riesenfeld algorithm for knots in arithmetic progression. For \( \beta \neq 1 \), the B-spline curve may be reparameterized via equation 12 to bring the knots into geometric progression. Since this reparameterization is translation, it does not affect the geometry of the B-spline curve. After applying the subdivision algorithm of section 2, equation 11 may be used to compute the new refined knot sequence.

References


