Sparse Filter Banks for Binary Subdivision Schemes

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Abstract

Multi-resolution analysis (MRA) produces a hierarchical basis for representing functions. This basis can be used to improve the efficiency of many algorithms for computing with those functions. Traditionally, these basis functions are translates and dilates of a single function. In this paper, we outline a generalization of MRA to those functions defined by binary subdivision schemes.

Although the mathematical underpinnings of MRA are somewhat involved, the resulting algorithms are quite simple. We start with a brief intuitive description of how the method can be applied to decompose the polyhedral object shown in Figure 1(a).

The main idea behind MRA is the decomposition of an object, in this case a polyhedron, into a low resolution part and a "detail" part. The low resolution part of the polyhedron in Figure 1(a) is shown in Figure 1(b). The vertices in (b) are computed as certain weighted averages of the vertices in (a). These weighted averages essentially implement a low pass filter denoted as **A**. The detail part consists of a collection of fairly abstract coefficients, called wavelet coefficients, that are also computed as weighted averages of the

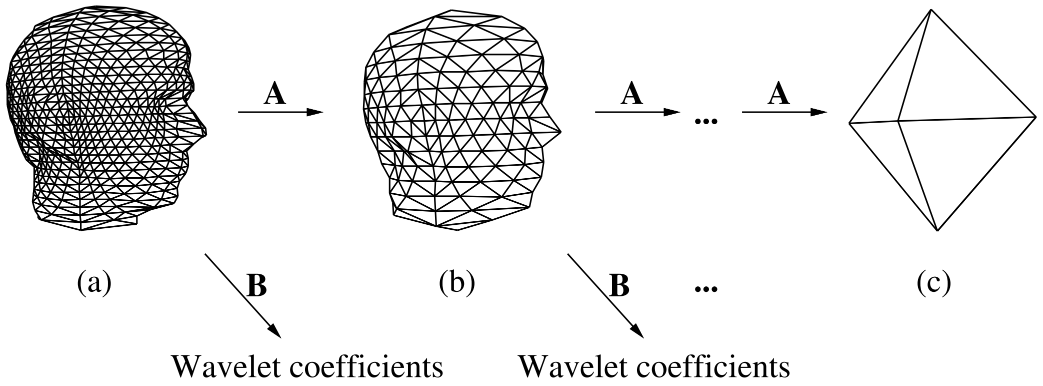


FIGURE 1. (a) Polyhedron in V^4, (b) Projection into V^3, (c) Projection into V^0.

vertices in (a), the weights forming a high-pass filter $\mathbf{B}$. The decomposition process, technically called analysis, can be used to further split (b) into an even lower resolution version and corresponding wavelet coefficients. This cascade of analysis steps is often referred to as a filter bank algorithm.

An interesting application of MRA is curve editing as demonstrated by Finkelstein and Salesin [6]. The idea is to allow for changes in the overall sweep of the curve by modifying broad-scale wavelet coefficients; fine-scale changes can similarly be made by modifying only fine-scale wavelet coefficients, as shown in figure 2.

The curve in part (a) can be decomposed into the low frequency, smooth curve in part (b) plus a high-frequency component. The user can edit the low frequency curve of part (b) and produce the new low frequency curve in part (c). Adding back the high frequency noise produces the final modified curve of part (d).

1 A matrix view of multi-resolution analysis

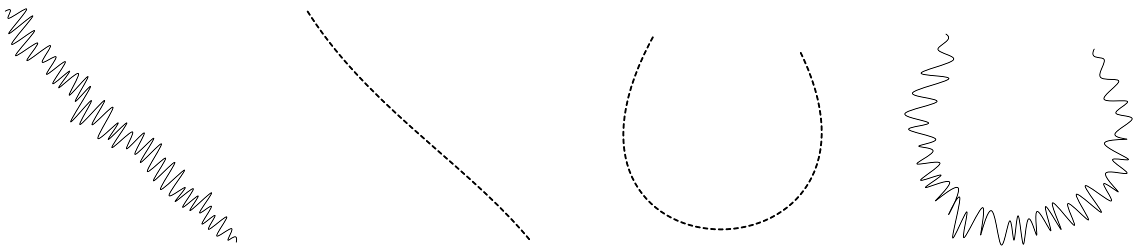
The theory of multi-resolution analysis can be formulated in many ways. Standard MRA considers translates and dilates of a single basis function. [5] gives a fairly informal introduction into this area using linear algebra and matrices. The advantage of using linear algebra is that many of the concepts of MRA can be defined without resort to translation and dilation. Under this approach, MRA can be described for arbitrary nested spline spaces.

Let $\Phi^j(t)$ denote a set of compactly supported basis functions, also known as *scaling functions*. The span of these basis function is a spline space, V^j,

$$V^j := \mathrm{Span}(\Phi^j(t)).$$

If these spline spaces are nested

$$V^0 \subset V^1 \subset \cdots,$$



(a) Original curve. (b) Overall sweep. (c) Modified sweep. (d) Modified curve.

FIGURE 2. Multi-resolution editing

then the resulting hierarchy of linear spaces are related by the matrix equation:

$$\Phi^j(t) = \Phi^{j+1}(t) S^j. \tag{1.1}$$

The matrix S^j is the *subdivision matrix* associated with level j.

Given this nesting, the fundamental problem in MRA is choosing a set of *wavelet functions* $\Psi^j(t)$ in V^{j+1} with the property that $\Phi^j(t)$ and $\Psi^j(t)$ together form a basis for V^{j+1}. In matrix form,

$$(\Phi^j(t)\Psi^j(t)) = \Phi^{j+1}(t) \begin{pmatrix} S^j & Q^j \end{pmatrix}. \tag{1.2}$$

If the span of the functions $\Psi^j(t)$ is denoted W^j, then

$$V^{j+1} = V^j + W^j.$$

This new basis is *hierarchical* since a function $f^{j+1}(t)$ in V^{j+1} can be expressed as

$$f^{j+1}(t) = f^j(t) + g^j(t)$$

where $f^j(t)$ is in V^j and $g^j(t)$ is in W^j.

If the inverse transformation is written in block matrix form

$$\Phi^{j+1}(t) = (\Phi^j(t)\Psi^j(t)) \begin{pmatrix} A^j \\ B^j \end{pmatrix} \tag{1.3}$$

then, the matrices A^j and B^j can be used to decompose a function $f^{j+1}(t)$ in V^{j+1} given by

$$f^{j+1}(t) = \sum_i f_i^{j+1} \phi_i^{j+1}(t) \tag{1.4}$$

into a lower resolution part in V^j plus a detail part in W^j

$$f^{j+1}(t) = \sum_i f_i^j \phi_i^j(t) + \sum_i g_i^j \psi_i^j(t).$$

Let F^j and G^j be the vector form of the coefficients f_i^j and the g_i^j, respectively. We now write Equation 1.4 in matrix form and substitute in equation 1.3:

$$\begin{aligned} f^{j+1}(t) &= \Phi^{j+1}(t)\ F^{j+1} \\ &= \left(\Phi^j(t)\ \Psi^j(t)\right) \begin{pmatrix} A^j \\ B^j \end{pmatrix} F^{j+1} \\ &= \Phi^j(t) A^j\ F^{j+1} + \Psi^j(t) B^j\ F^{j+1} \end{aligned}$$

and therefore

$$F^j = A^j\ F^{j+1} \qquad\qquad G^j = B^j\ F^{j+1}.$$

The matrices A^j and B^j are *analysis filters*. They are used to convert a high-resolution set of coefficients F^{j+1} into a low-resolution set of coefficients F^j plus detail coefficients G^j. Of course, the analysis filters A^{j-1} and B^{j-1} can now be applied to F^j to yield F^{j-1} and G^{j-1} and so on.

Conversely, F^{j+1} can be recovered from F^j and G^j using equation 1.2:

$$F^{j+1} = S^j\ F^j + Q^j\ G^j.$$

The matrices S^j and Q^j are *synthesis filters*. They are used to convert low-resolution coefficients and detail coefficients into a high-resolution set of coefficients.

2 Filter banks for binary subdivision schemes

The subdivision matrices arising from compactly supported, binary subdivision schemes have a very special structure. A bi-infinite matrix S is 2-*slanted* if there exists an n such that

$$S_{ik} = 0$$

for all $i \notin \{2k-n, ..., 2k+n\}$. The *column height* of S is $2n+1$ where n takes on the minimum value satisfying the previous definition. If the matrices S^j arise from a compactly supported, binary subdivision scheme, then the matrices S^j are 2-slanted.

Given a sequence of 2-slanted subdivision matrices S^j (defining the scaling function $\Phi^j(t)$), this paper focuses on the problem of finding a sequence of 2-slanted synthesis filters Q^j (defining the wavelets $\Psi^j(t)$) whose inverse analysis filters A^j and B^j are sparse. The next section describes a fast, local construction for generating Q^j, A^j, and B^j from S^j.

An initial pair of synthesis filters $\begin{pmatrix} S^j & Q^j \end{pmatrix}$ and their corresponding inverse analysis filters $\begin{pmatrix} A^j \\ B^j \end{pmatrix}$ satisfy the relation

$$\begin{pmatrix} S^j & Q^j \end{pmatrix} \begin{pmatrix} A^j \\ B^j \end{pmatrix} = I.$$

Any other sets of synthesis and analysis filters can be expressed in terms of these initial filters using *lifting* [7]. The lifted synthesis filters take the form $\begin{pmatrix} S^j & Q^j + S^j\alpha^j \end{pmatrix}$. The lifted analysis filters have the form $\begin{pmatrix} A^j - \alpha^j B^j \\ B^j \end{pmatrix}$. Note that the lifted filters remain inverses:

$$\begin{pmatrix} S^j & Q^j + S^j\alpha^j \end{pmatrix} \begin{pmatrix} A^j - \alpha^j B^j \\ B^j \end{pmatrix} = I.$$

If α^j is chosen to be banded, the lifted filters are also sparse.

Of course, sparseness is not sufficient to guarantee that the resulting filter banks generate a satisfactory MRA. Applying a sequence of unlifted analysis filters can greatly magnify any numerical errors present in an initial set of coefficients. Typically, the initial filter banks are lifted such that applying the resulting sequence of analysis filters is numerically stable. For a detailed description of this problem, we refer the reader to [2].

3 A construction for bi-infinite subdivision matrices

Let S^j denote a bi-infinite 2-slanted subdivision matrix with column height $2n+1$. We describe a local construction that produces a 2-slanted Q^j of column height $2n-1$ whose inverse filters A^j and B^j are the transpose of 2-slanted matrices with column heights $2n-1$ and $2n+1$, respectively. Since this construction is independent of j, the index may be dropped without loss of generality.

3.1 Computation of Q

Our goal is to compute c_k, the non-zero portion of the kth column of Q, spanning rows $2k+2-n$ to $2k+n$. To this end, we define two adjacent submatrices of S spanning these rows,

- L_k spanning columns $k-n+1$ to k and,
- R_k spanning columns $k+1$ to $k+n$.

We claim that if c_k lies in the intersection of the column spaces of L_k and R_k, then the resulting Q, A, and B have the desired sparseness. We proceed in two parts. First, we give a direct construction for c_k. Then, we prove that A, and B have the desired sparseness.

After rejoining L_k and R_k, the matrix $(L_k \ R_k)$ has size $(2n-1)\times 2n$. Therefore, there must exist a vector in its nullspace of the form

$$(L_k \ R_k)\begin{pmatrix} l_k \\ -r_k \end{pmatrix} = 0.$$

Setting $c_k = L_k l_k = R_k r_k$ yields a vector in the column spaces of both L_k and R_k.

In terms of Q, the following lemma follows directly.

Lemma 3.1 *Restricted to rows $2k+2-n$ to $2k+n$,*

$$\begin{aligned} Q_k &\in span\{S_j | j \leq k\} \\ &\in span\{S_j | j > k\}. \end{aligned}$$

where Q_k and S_k are kth columns of Q and S, respectively.

For example, a typical portion of the subdivision matrix for a uniform cubic B-spline has the form:

$$S = \begin{pmatrix} 6 & 1 & 0 & 0 & 0 \\ 4 & 4 & 0 & 0 & 0 \\ 1 & 6 & 1 & 0 & 0 \\ 0 & 4 & 4 & 0 & 0 \\ 0 & 1 & 6 & 1 & 0 \\ 0 & 0 & 4 & 4 & 0 \\ 0 & 0 & 1 & 6 & 1 \\ 0 & 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 1 & 6 \end{pmatrix} \quad Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since the scheme is uniform, the matrix $(L_k \ R_k)$ is the same for all k and satisfies

$$\begin{pmatrix} 1 & 6 & 1 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 1 & 6 & 1 \end{pmatrix} \begin{pmatrix} -5 \\ 1 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore, the vector c_k can be expressed as

$$\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 0 & 4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -5 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 4 & 0 \\ 6 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -5 \end{pmatrix}.$$

Note that if $(L_k R_k)$ has full rank, then c_k is unique up to multiplication by a constant. We defer the choice of this constant till later in this section.

Under certain conditions on S, c_k may be identically zero. In this case, the construction for Q fails and the inverse filters A and B do not exist. Conversely, if the inverse filters A and B exist, then Q exists and $c_k \neq 0$. We next consider appropriate conditions on S to ensure the existence of A and B.

3.2 Computation of A

Due to the sparsity of S, any set of $2n+1$ consecutive rows of S must be linearly dependent. If all sets of $2n$ consecutive rows of S are linearly independent, then S has *maximal row rank*. As the following constructions will show, maximal row rank ensures that the inverse filters A and B exist.

First, note that the analysis filter A should satisfy $AS = I$ and $AQ = 0$ where S and Q are the synthesis filters. We next show that A can easily be computed from S on a row by row basis. Let A_k denote kth row of A.

Theorem 3.1 *If S has maximal row rank, then there exists unique A_k supported over columns $2k-n+1$ to $2k-1+n$ such that*

$$\begin{aligned} A_k S &= e_k, \\ A_k Q &= 0, \end{aligned}$$

where e_k is a vector whose kth entry is one and zero elsewhere.

> **Proof** Consider the portion of S spanning rows $2k-n+1$ to $2k-1+n$. These $2n-1$ rows are supported over $2n-1$ columns, numbered $k-n+1$ to $k+n-1$. These rows are linearly independent since S has maximal row rank. Thus, there exists a unique row vector A_k supported over columns $2k-n+1$ to $2k-1+n$ such that
>
> $$A_k S = e_k.$$
>
> Recall that Q_j is the jth column of Q. If $j < k$, then Q_j lies in the span of columns S_i where $i \leq j < k$ by lemma 3.1. Since $A_k S_i = 0$ for $i < k$, $A_k Q_j$ must also be zero. If $j \geq k$, then Q_j lies in the span of columns S_i where $i > j \geq k$. Since $A_k S_i = 0$ for $i > k$, $A_k Q_j$ is again zero. $\square$

Continuing the cubic B-spline example, the non-zero portion of the rows of A are generated by the following equation

$$\begin{pmatrix} -1 & 4 & -1 \end{pmatrix} \begin{pmatrix} 4 & 4 & 0 \\ 1 & 6 & 1 \\ 0 & 4 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 16 & 0 \end{pmatrix}.$$

For example, row four of A satisfies

$$\begin{pmatrix} 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 6 & 1 & 0 & 0 & 0 \\ 4 & 4 & 0 & 0 & 0 \\ 1 & 6 & 1 & 0 & 0 \\ 0 & 4 & 4 & 0 & 0 \\ 0 & 1 & 6 & 1 & 0 \\ 0 & 0 & 4 & 4 & 0 \\ 0 & 0 & 1 & 6 & 1 \\ 0 & 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 1 & 6 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 16 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that this row annihilates Q.

$$\begin{pmatrix} 0\ 0\ 0\ -1\ 4\ -1\ 0\ 0\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \end{pmatrix}.$$

3.3 Computation of B

The analysis filter B satisfies $BS = 0$ and $BQ = I$. We claim that B can also be computed from S a row at a time. Let B_k denote the kth row of B.

Theorem 3.2 *If S has maximal row rank, then there exists B_k (unique up to multiplication by a constant) supported over columns $2k-n+1$ to $2k+n+1$ such that*

$$\begin{aligned} B_k S &= 0, \\ B_k Q &= \hat{e}_k \end{aligned}$$

where $\hat{e}_k$ is a multiple of e_k (possibly zero).

Proof Consider the portion of S spanning rows $2k-n+1$ to $2k+n+1$. These $2n+1$ rows are supported over $2n$ columns, numbered $k-n+1$ to $k+n$. Therefore, these rows are linearly dependent and there exists a row vector B_k supported over columns $2k-n+1$ to $2k+n+1$ such that

$$B_k S = 0. \tag{3.1}$$

Since the $2n+1$ rows have rank $2n$ (due to S having maximal row rank), this vector is unique up to multiplication by a constant.

By lemma 3.1, the restriction of Q_j to rows $2j-n+2$ to $2j+n$ can be written as a linear combination of similarly restricted columns of S. If $j < k$, then the restriction of Q_j to rows $2k-n+1$ to $2k+n+1$ can also be expressed as a combination of similarly restricted columns of S. (Any new rows introduced are all zero.) By equation 3.1, the product of these columns of S with B_k is zero. Therefore, $B_k Q_j = 0$ for $j < k$.

A similar proof shows that $B_k Q_j = 0$ if $j > k$. $\square$

To complete the construction of B, we must show that $B_k Q_k = 1$. The next theorem shows that if S has maximal row rank, then $B_k Q_k$ is not equal to zero. Recall that both B_k and Q_k were only unique up to multiplication by a constant. In fact, choosing the constant for Q_k fixes the constant for B_k such that $B_k Q_k = 1$.

Theorem 3.3 *Let B_k be defined as in the previous theorem. If S has maximal row rank, then*

$$B_k Q_k \neq 0.$$

Proof Inserting the column Q_k between columns k and $k+1$ of S and extracting rows $2k-n+1$ to $2k+n+1$ yields a block matrix of the form:

$$\begin{pmatrix} u_k & 0 & 0 \\ L_k & c_k & R_k \\ 0 & 0 & v_k \end{pmatrix} \tag{3.2}$$

where u_k is the non-zero portion of row $2k-n+1$ of S and v_k is the non-zero portion of row $2k+n+1$ of S. $B_k Q_k = 0$ if and only if the nullspace of this matrix is non-empty. By construction of Q_k, the nullspace of $(L_k\ c_k\ R_k)$ consists of exactly two vectors, $(l_k\ -1\ 0)$ and $(0\ -1\ r_k)$.

By the construction of l_k and r_k, the determinant of the matrix

$$\begin{pmatrix} u_k & 0 \\ L_k & R_k \end{pmatrix}$$

is exactly $u_k l_k$. Since this matrix is a submatrix of S and S has maximal row rank, the determinant $u_k l_k$ is not zero. Therefore, $(l_k\ -1\ 0)$ cannot be in the nullspace of $(u_k\ 0\ 0)$. A similar argument show that $(0\ -1\ r_k)$ is not in the nullspace of $(0\ 0\ v_k)$. Therefore, the nullspace of matrix 3.2 is empty and $B_k Q_k \neq 0$. $\square$

Completing our running cubic example, the rows of B satisfy the equation

$$\begin{pmatrix} 1 & -4 & 6 & -4 & 1 \end{pmatrix} \begin{pmatrix} 4 & 4 & 0 & 0 \\ 1 & 6 & 1 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 1 & 6 & 1 \\ 0 & 0 & 4 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}.$$

Multiplying this row with Q yields a multiple of the basis vector

$$\begin{pmatrix} 1 & -4 & 6 & -4 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 16 & 0 \end{pmatrix}.$$

4 A construction for infinite subdivision matrices

The previous section considered the case of a bi-infinite 2-slanted matrix S. Such matrices arise from binary subdivision schemes applied to the interior of a curve. The matrix defined by a binary subdivision scheme at the endpoint of a curve is only infinite and has a slightly different structure.

If the subdivision scheme forces the kth derivative at the endpoint of a curve to depend on at most the last $k+1$ coefficients of the curve, then the subdivision S is lower triangular in addition to being 2-slanted. If the rows and columns of S are numbered starting at zero, then these two conditions are equivalent to

$$S_{ik} = 0$$

for all $i \notin \{max(k, 2k-n), ..., 2k+n\}$

For example, one typical set of subdivision rules at the endpoint of a cubic B-spline curve has the form:

$$S = \begin{pmatrix} 8 & 0 & 0 & 0 & . \\ 4 & 4 & 0 & 0 & . \\ 1 & 6 & 1 & 0 & . \\ 0 & 4 & 4 & 0 & . \\ 0 & 1 & 6 & 1 & . \\ 0 & 0 & 4 & 4 & . \\ . & . & . & . & . \end{pmatrix}.$$

This section sketches a simple modification to the previous construction for matrices S of this form.

If S has column height $2n+1$, then we construct the synthesis filter Q on a column by column basis. If $k \geq n$, then construction for c_k given in the previous section can be applied. If $0 \leq k < n$, then we modify the construction of c_k slightly.

We let c_k, the non-zero portion of the kth column of Q, span only rows $k+1$ to $2k+n$. The matrix L_k is chosen to span columns 0 to k and the matrix R_k remains unchanged. Again, the vector c_k is chosen to lie in the intersection of the column spaces of L_k and R_k. Since, the composite matrix $(L_k\ R_k)$ has size $(n+k) \times ((n+k+1)$, the same construction can be used to compute c_k.

For example, the matrix $(L_0\ R_0)$ satisfies

$$\begin{pmatrix} 4 & 4 & 0 \\ 1 & 6 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Therefore, the vector c_0 can be expressed as

$$\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 6 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -5 \end{pmatrix}.$$

The constructions of the analysis filters A and B proceeds in a similar manner. We leave the specifics as an exercise for the interested reader.

5 Relation to previous work

The method described in the previous section unifies several previously known methods and provides a simple, general framework for creating sparse filter banks.

If the $2k$th row of S is exactly e_k for all k, then the subdivision matrix S defines an *interpolating* subdivision scheme. Previous methods for interpolating schemes such as [3] and [7] have suggested using an initial synthesis filter Q where Q_k is exactly e_{2k}. For interpolating schemes, the method of the previous section also produces columns Q_k that are multiples of e_{2k}. Therefore, this method can be viewed as a generalization of previous methods for interpolating schemes to the more general approximating case.

Cohen, Daubechies and Feaveau [1] describe a method for creating biorthogonal wavelets. One component of this method is a construction that produces sparse filters banks if all of the columns of S are shifts of a single column (i.e. the subdivision scheme is *uniform*). Again, our method exactly reproduces these filter banks for the uniform case in addition to working in the non-uniform case.

Finally, Dahmen and Micchelli [4] describe a method for creating banded matrices with banded inverses. Cast in our framework, this method produces essentially the same filter banks as our method. However, we feel our construction is more concise and explicitly demonstrates the relationship of the filters Q, A, and B to the original subdivision matrix S.

Acknowledgements

The author would like to thank Henrik Weimer for his help in preparing the paper.

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