COMP 481: Automata, Formal Languages, and Computability
Spring 2009
Homework Assignment #7 (Due date: 31 March 2009)

1. For each of the following languages, state whether each language is (I) decidable ($\in$D), (II) semidecidable but not decidable ($\in$SD$-$D), or (III) not semidecidable ($\notin$SD). Prove your answer.

   - $L_1 = \{\langle M\rangle | M$ is a TM and there exists an input on which M halts in less than $|\langle M\rangle|$ steps$\}$.
   - $L_2 = \{\langle M\rangle | M$ is a TM that accepts all even numbers$\}$.
   - $L_3 = \{\langle M\rangle | M$ is a TM and $L(M)$ is infinite$\}$.
   - $L_4 = \{\langle M\rangle | M$ is a TM and $L(M)$ is countable$\}$.
   - $L_5 = \{\langle M_1, M_2\rangle | M_1$ and M_2 are two TMs, and $\varepsilon \in L(M_1) \cap L(M_2)\}$.
   - $L_6 = \{\langle M_1, M_2\rangle | M_1$ and M_2 are two TMs, and $\varepsilon \in L(M_1) \setminus L(M_2)\}$.
   - $L_7 = \{\langle M_1, M_2\rangle | L(M_1) \leq L(M_2)\}$, where $L_1 \leq L_2$ denotes that L_1 reduces to L_2.
   - $L_8 = \{\langle M\rangle | M$ is a DFA and M accepts some string of the form ww^R for some $w \in \{a, b\}^*\}$.

2. A language L is **SD-Complete** if:

   - $L \in SD$, and
   - $L' \leq L$ for all $L' \in SD$.

   Consider the following two languages::

   $$L_{\Sigma^*} = \{\langle M\rangle |\ L(M) = \Sigma^*\}$$
   $$HP = \{\langle M, w\rangle |\ M \text{ halts on } w\}$$

   (a) Is L_{Σ^*} SD-Complete or not? Prove your answer.

   (b) Is HP SD-Complete or not? Prove your answer.

3. Prove or disprove: there exists an undecidable unary language (a unary language is a subset of $\{1\}^*$).

4. Let A and B be two disjoint languages. Say that language C *separates* A and B if $A \subseteq C$ and $B \subseteq \overline{C}$. Show that any two disjoint co-semi-decidable[1] languages are separable by some decidable language.

5. Let L be the language containing only the single string s, where

[1] A language L is co-semi-decidable if $\overline{L}$ is semi-decidable.

$$s = \begin{cases} 0 & if \quad God\ does\ not\ exist \\ 1 & if \quad God\ exists \end{cases}$$

Is L decidable? Why or why not? (Note that the answer does not depend on your religious convictions.)

6. If $A \leq B$ and B is a regular language, does that imply that A is a regular language?

7. Let L_1, L_2 be two decidable languages, and let L be a language such that $L_1 \subseteq L \subseteq L_2$. Is L decidable or not? Prove your answer.

8. Suppose there are four languages A, B, C, and D. Each of the languages may or may not be semidecidable. However, we know the following about them:

   - There is a reduction from A to B.
   - There is a reduction from B to C.
   - There is a reduction from D to C.

   Below are four statements. Indicate whether each one is

   (a) CERTAIN to be true, regardless of what problems A through D are.
   (b) MAYBE true, depending on what A through D are.
   (c) NEVER true, regardless of what A through D are.

   Please, justify your answer!

   (a) A is semidecidable but not decidable, and C is decidable.
   (b) A is not decidable, and D is not semidecidable.
   (c) If C is decidable, then the complement of D is decidable.
   (d) If C is semidecidable, then $B \cap D$ is semidecidable.

9. A *Turing machine with doubly infinite tape* is similar to an ordinary Turing machine, but its tape is infinite to the left as well as to the right. The tape is initially filled with blanks except for the portion that contains the input. Computation is defined as usual except that the head never encounters an end to the tape as it moves leftward. Show that this type of Turing machine recognizes the case of Turing-recocgnizable (semi-decidable, or recursively enumerable) languages.

10. A *Turing machine with stay put instead of left* is similar to an ordinary Turing machine, but the transition function has the form

$$\delta : K \times \Gamma \to K \times \Gamma \times \{\to, \updownarrow\},$$

where $\updownarrow$ indicates that the Turing machine's reading head stays put (does not move left or right) after executing the transition. Show that this Turing machine variant is <u>not</u> equivalent to the usual version. What class of languages do these machines recognize?

11. Show that every infinite language in SD has an infinite decidable subset.