

COMP 481: Automata, Formal Languages, and Computability

Spring 2009

Homework Assignment #8 (Due date: 9 April 2009)

1. Let A be a language in SD consisting of descriptions of Turing machines, $\{\langle M_1 \rangle, \langle M_2 \rangle, \dots\}$, where every M_i is a decider. Prove that some decidable language L is not decided by any decider M_i whose description appears in A . (hint: you may find it helpful to consider an enumerator for A).

2. Let $T = \{\langle M \rangle : M \text{ is a TM that accepts } w^R \text{ whenever it accepts } w\}$. Show that T is undecidable.

3. Let $\Gamma = \{0, 1, \sqcup\}$ be the tape alphabet for all TMs in this problem. Define the *busy beaver function*

$$BB : \mathbb{N} \rightarrow \mathbb{N}$$

as follows. For each $k \in \mathbb{N}$, consider all k -state TMs that halt when started with a blank tape. Let $BB(k)$ be the maximum number of 1s that remain on the tape among all of these machines. Show that BB is not a computable function.

4. Define $A_{TM} = \{\langle M, w \rangle : M \text{ is a TM that accepts } w\}$. Prove that language A is in SD if and only if $A \leq_m A_{TM}$.

5. Show that A is decidable if and only if $A \leq_m 0^*1^*$.

6. Let

$$f(x) = \begin{cases} 3x + 1 & \text{for odd } x \\ x/2 & \text{for even } x \end{cases}$$

for any natural number x . If you start with an integer x and iterate f , you obtain a sequence

$$x, f(x), f(f(x)), \dots$$

Stop if you ever hit 1. For example, if $x = 17$, you get the sequence

$$17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1.$$

Extensive computer tests have shown that every starting point between 1 and a large positive integer gives a sequence that ends in 1. But, the question of whether all positive starting points end up at 1 is unsolved.

Suppose that A_{TM} (see Problem 4) were decidable by a TM H . Use H to describe a TM that is guaranteed to state the answer to the problem; i.e., decides whether the sequence starting at a number x would end up at 1.

7. Let $S = \{\langle M \rangle : M \text{ is a TM and } L(M) = \{\langle M \rangle\}\}$. Show that neither S nor \bar{S} is in SD.

8. Prove, using Rice's theorem, that the following languages are not decidable. (See the solution to Problem 5.30(a) in the textbook for an example of how to use Rice's theorem.)

- $L_1 = \{\langle M \rangle \mid \exists x, |x| \equiv_5 1, \text{ and } x \in L(M)\}$.
- $L_2 = \{\langle M \rangle \mid M \text{ is a TM, and } M \text{ accepts all palindromes}\}$.
- $L_3 = \{\langle M \rangle \mid M \text{ does not accept any string } w \text{ such that } 001 \text{ is a prefix of } w\}$.

9. Recall the following definition: A grammar G computes a function f iff for all $u, v \in \Sigma^*$,

$$SuS \Rightarrow_G^* v \text{ iff } f(u) = v.$$

For each of the following functions, show a grammar that computes it. In the functions f_1 and f_2 , both n and $f(n)$ are unary representations of natural numbers.

- $f_1(n) = 3n + 5$.
- $f_2(n) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod{3} \\ 11 & \text{if } n \equiv 1 \pmod{3} \\ 111 & \text{if } n \equiv 2 \pmod{3} \end{cases}$
- $f_3(a_1 a_2 \dots a_k) = a_1 a_1 a_2 a_2 \dots a_k a_k$, where each a_i is in the alphabet $\{a, b\}$.
- $f_4(w) = ww$, for every $w \in \{a, b\}^*$.

10. Give grammars that generate the following languages.

- $L_1 = \{x\#w \mid x, w \in \{a, b\}^* \text{ and } x \text{ is a substring of } w\}$.
- $L_2 = \{w \in \{a, b, c\}^* \mid \#_a(w) \geq \#_b(w) \geq \#_c(w)\}$.
- $L_3 = \{a^n b^n c a^n b^n \mid n > 0\}$.
- $L_4 = \{a^n b^{2n} c^{3n} \mid n \geq 0\}$.
- $L_5 = \{a^n b^{n+m} c^m d^n \mid m, n \geq 0\}$.