

# COMP 481: Automata, Formal Languages, and Computability

Spring 2008

## Homework Assignment #9 (Due date: 21 April 2008)

1. (a) Describe a TM that computes the sum of two numbers represented in binary.  
(b) Describe a TM that converts binary numbers to their unary representation.
2. Suppose  $L_1, \dots, L_k$  form a partition of  $\Sigma^*$ . Prove that if  $L_i \in RE$ , for all  $1 \leq i \leq k$ , then  $L_i \in R$  as well.
3. Prove that D is closed under
  - (a) union
  - (b) concatenation
  - (c) Kleene start
  - (d) intersection
4. Prove that SD is closed under
  - (a) union
  - (b) concatenation
  - (c) Kleene start
  - (d) intersection
5. For each of the following languages, state whether each language is (I) decidable ( $\in D$ ), (II) semidecidable but not decidable ( $\in SD-D$ ), or (III) not semidecidable ( $\notin SD$ ). Prove your answer.
  - $L_1 = \{\langle M \rangle \mid M \text{ is a TM and there exists an input on which } M \text{ halts in less than } |\langle M \rangle| \text{ steps}\}$ .
  - $L_2 = \{\langle M \rangle \mid M \text{ is a TM that accepts all even numbers}\}$ .
  - $L_3 = \{\langle M \rangle \mid M \text{ is a TM and } L(M) \text{ is infinite}\}$ .
  - $L_4 = \{\langle M \rangle \mid M \text{ is a TM and } L(M) \text{ is countable}\}$ .
  - $L_5 = \{\langle M_1, M_2 \rangle \mid M_1 \text{ and } M_2 \text{ are two TMs, and } \varepsilon \in L(M_1) \cap L(M_2)\}$ .
  - $L_6 = \{\langle M_1, M_2 \rangle \mid M_1 \text{ and } M_2 \text{ are two TMs, and } \varepsilon \in L(M_1) \setminus L(M_2)\}$ .
  - $L_7 = \{\langle M_1, M_2 \rangle \mid L(M_1) \leq L(M_2)\}$ , where  $L_1 \leq L_2$  denotes that  $L_1$  reduces to  $L_2$ .
  - $L_8 = \{\langle M \rangle \mid M \text{ is a DFA and } M \text{ accepts some string of the form } ww^R \text{ for some } w \in \{a, b\}^*\}$ .
6. A language  $L$  is **SD-Complete** if:

- $L \in SD$ , and
- $L' \leq L$  for all  $L' \in SD$ .

Consider the following two languages::

$$L_{\Sigma^*} = \{\langle M \rangle \mid L(M) = \Sigma^*\}$$

$$HP = \{\langle M, w \rangle \mid M \text{ halts on } w\}$$

- (a) Is  $L_{\Sigma^*}$  SD-Complete or not? Prove your answer.
- (b) Is  $HP$  SD-Complete or not? Prove your answer.
7. Prove or disprove: there exists an undecidable unary language (a unary language is a subset of  $\{1\}^*$ ).
8. Let  $A$  and  $B$  be two disjoint languages. Say that language  $C$  *separates*  $A$  and  $B$  if  $A \subseteq C$  and  $B \subseteq \overline{C}$ . Show that any two disjoint co-Turing-recognizable<sup>1</sup> languages are separable by some decidable language.
9. Let  $L$  be the language containing only the single string  $s$ , where

$$s = \begin{cases} 0 & \text{if } \text{God does not exist} \\ 1 & \text{if } \text{God exists} \end{cases}$$

Is  $L$  decidable? Why or why not? (Note that the answer does not depend on your religious convictions.)

10. If  $A \leq B$  and  $B$  is a regular language, does that imply that  $A$  is a regular language?
11. Let  $L_1, L_2$  be two decidable languages, and let  $L$  be a language such that  $L_1 \subseteq L \subseteq L_2$ . Is  $L$  decidable or not? Prove your answer.
12. Suppose there are four languages  $A, B, C$ , and  $D$ . Each of the languages may or may not be semidecidable. However, we know the following about them:
- There is a reduction from  $A$  to  $B$ .
  - There is a reduction from  $B$  to  $C$ .
  - There is a reduction from  $D$  to  $C$ .

Below are four statements. Indicate whether each one is

- (a) CERTAIN to be true, regardless of what problems  $A$  through  $D$  are.
- (b) MAYBE true, depending on what  $A$  through  $D$  are.
- (c) NEVER true, regardless of what  $A$  through  $D$  are.

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<sup>1</sup>A language  $L$  is co-Turing-recognizable if  $\overline{L}$  is Turing-recognizable.

**Please, justify your answer!**

- (a)  $A$  is semidecidable but not decidable, and  $C$  is decidable.
  - (b)  $A$  is not decidable, and  $D$  is not semidecidable.
  - (c) If  $C$  is decidable, then the complement of  $D$  is decidable.
  - (d) If  $C$  is semidecidable, then  $B \cap D$  is semidecidable.
13. A Turing machine with doubly infinite tape is similar to an ordinary Turing machine, but its tape is infinite to the left as well as to the right. The tape is initially filled with blanks except for the portion that contains the input. Computation is defined as usual except that the head never encounters an end to the tape as it moves leftward. Show that this type of Turing machine recognizes the case of Turing-recognizable (semi-decidable, or recursively enumerable) languages.
14. A Turing machine with stay put instead of left is similar to an ordinary Turing machine, but the transition function has the form

$$\delta : K \times \Gamma \rightarrow K \times \Gamma \times \{\rightarrow, \uparrow\},$$

where  $\uparrow$  indicates that the Turing machine's reading head stays put (does not move left or right) after executing the transition. Show that this Turing machine variant is not equivalent to the usual version. What class of languages do these machines recognize?

15. Show that every infinite language in SD has an infinite decidable subset.
16. Let  $A$  be a language in SD consisting of descriptions of Turing machines,  $\{\langle M_1 \rangle, \langle M_2 \rangle, \dots\}$ , where every  $M_i$  is a decider. Prove that some decidable language  $L$  is not decided by any decider  $M_i$  whose description appears in  $A$ . (hint: you may find it helpful to consider an enumerator for  $A$ ).
17. Let  $T = \{\langle M \rangle : M \text{ is a TM that accepts } w^R \text{ whenever it accepts } w\}$ . Show that  $T$  is undecidable.
18. Let  $\Gamma = \{0, 1, \sqcup\}$  be the tape alphabet for all TMs in this problem. Define the *busy beaver function*

$$BB : \mathbb{N} \rightarrow \mathbb{N}$$

as follows. For each  $k \in \mathbb{N}$ , consider all  $k$ -state TMs that halt when started with a blank tape. Let  $BB(k)$  be the maximum number of 1s that remain on the tape among all of these machines. Show that  $BB$  is not a computable function.

19. Define  $A_{TM} = \{\langle M, w \rangle : M \text{ is a TM that accepts } w\}$ . Prove that language  $A$  is in SD if and only if  $A \leq_m A_{TM}$ .
20. Show that  $A$  is decidable if and only if  $A \leq_m 0^*1^*$ .

21. Let

$$f(x) = \begin{cases} 3x + 1 & \text{for odd } x \\ x/2 & \text{for even } x \end{cases}$$

for any natural number  $x$ . If you start with an integer  $x$  and iterate  $f$ , you obtain a sequence

$$x, f(x), f(f(x)), \dots$$

Stop if you ever hit 1. For example, if  $x = 17$ , you get the sequence

$$17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1.$$

Extensive computer tests have shown that every starting point between 1 and a large positive integer gives a sequence that ends in 1. But, the question of whether all positive starting points end up at 1 is unsolved.

Suppose that  $A_{TM}$  (see Problem 19) were decidable by a TM  $H$ . Use  $H$  to describe a TM that is guaranteed to state the answer to the problem; i.e., decides whether the sequence starting at a number  $x$  would end up at 1.

22. Let  $S = \{\langle M \rangle : M \text{ is a TM and } L(M) = \{\langle M \rangle\}\}$ . Show that neither  $S$  nor  $\bar{S}$  is in SD.

23. Prove, using Rice's theorem, that the following languages are not decidable. (See the solution to Problem 5.30(a) in the textbook for an example of how to use Rice's theorem.)

- $L_1 = \{\langle M \rangle \mid \exists x, |x| \equiv_5 1, \text{ and } x \in L(M)\}$ .
- $L_2 = \{\langle M \rangle \mid M \text{ is a TM, and } M \text{ accepts all palindromes}\}$ .
- $L_3 = \{\langle M \rangle \mid M \text{ does not accept any string } w \text{ such that } 001 \text{ is a prefix of } w\}$ .

24. Recall the following definition: A grammar  $G$  computes a function  $f$  iff for all  $u, v \in \Sigma^*$ ,

$$SuS \Rightarrow_G^* v \text{ iff } f(u) = v.$$

For each of the following functions, show a grammar that computes it. In the functions  $f_1$  and  $f_2$ , both  $n$  and  $f(n)$  are unary representations of natural numbers.

- $f_1(n) = 3n + 5$ .
- $f_2(n) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod{3} \\ 11 & \text{if } n \equiv 1 \pmod{3} \\ 111 & \text{if } n \equiv 2 \pmod{3} \end{cases}$
- $f_3(a_1 a_2 \dots a_k) = a_1 a_1 a_2 a_2 \dots a_k a_k$ , where each  $a_i$  is in the alphabet  $\{a, b\}$ .
- $f_4(w) = ww$ , for every  $w \in \{a, b\}^*$ .

25. Give grammars that generate the following languages.

- $L_1 = \{x\#w \mid x, w \in \{a, b\}^* \text{ and } x \text{ is a substring of } w\}$ .
- $L_2 = \{w \in \{a, b, c\}^* \mid \#_a(w) \geq \#_b(w) \geq \#_c(w)\}$ .
- $L_3 = \{a^n b^n c a^n b^n \mid n > 0\}$ .
- $L_4 = \{a^n b^{2n} c^{3n} \mid n \geq 0\}$ .
- $L_5 = \{a^n b^{n+m} c^m d^n \mid m, n \geq 0\}$ .