

COMP 481: Automata, Formal Languages, and Computability

Spring 2008

Solutions to Homework Assignment #1

1. (a) This statement is false only when $L = \emptyset$, since if this is the case, then $L \subseteq LL$ but $\varepsilon \notin L$. However, it is true for every $L \neq \emptyset$, and here is the proof. We need to prove two parts:
- $L \subseteq LL \Rightarrow \varepsilon \in L$. Let $x \in L$ be a string such that for every string $y \in L$, we have $|y| \geq |x|$. In other words, x is a “shortest” string in L . Since $L \subseteq LL$, then $x \in LL$, which in turn means $x = uv$, where $u, v \in L$. However, since x is a shortest string, this is only possible only if at least one of the two strings u and v equals ε . Hence, $\varepsilon \in L$.
 - $\varepsilon \in L \Rightarrow L \subseteq LL$. If $\varepsilon \in L$, then $LL = (\{\varepsilon\} \cup L)L = \{\varepsilon\}L \cup LL = L \cup LL$. Hence, $L \subseteq LL$.

Therefore, the claim is true.

- (b) True. We need to prove that $(\Sigma^*)^* \subseteq \Sigma^*$ and that $\Sigma^* \subseteq (\Sigma^*)^*$. The first part is trivial since every language is a subset of or equal to Σ^* . For the second part, let $x \in \Sigma^*$. Since $(\Sigma^*)^* = \{\varepsilon\} \cup \Sigma^* \cup (\Sigma^*)^2 \cup \dots$, it follows that $x \in (\Sigma^*)^*$. Therefore, the claim is true.
- (c) False. L^* is infinite if and only if neither $L = \{\varepsilon\}$ nor $L = \emptyset$.
2. (a) $A \oplus B = \{0000\}$ and $A \ominus B = \{\varepsilon\}$.
- (b) If both A and B are finite, then both $A \oplus B$ and $A \ominus B$ are finite. Let $A = \{a^{m_1}, \dots, a^{m_k}\}$ and $B = \{b^{n_1}, \dots, b^{n_p}\}$, where $m_i, n_j \in \mathbb{N}$, for all $1 \leq i \leq k$ and $1 \leq j \leq p$. Then, $|A \oplus B| \leq kp$ and $|A \ominus B| \leq kp$. Hence, both languages are finite.
3. (a) $\text{SplitP}(L) = \{ab, aa, bb, aaaa, aaab, aabb, abbb, bbbb\}$.
 $\text{SplitM}(L) = \{a^i b^j : i - j = 2 \text{ or } i - j = 4\}$.
- (b) If L is finite, then $\text{SplitP}(L)$ is finite as well, since every string $0^m \in L$ at most contributes $m + 1$ strings to $\text{SplitP}(L)$.
 If $L = \emptyset$, then $\text{SplitM}(L)$ is finite (\emptyset as well). Otherwise, $\text{SplitM}(L)$ is infinite, since L contains at least one string 0^m , $m \geq 0$, which contributes an infinite number of strings $a^i b^j$, $i - j = m$, to $\text{SplitM}(L)$.
4. We only give the ideas here, but not the actual state diagrams; you are supposed to draw them.
- (a) This language can be recognized with a DFA that has five states, $Q = \{q_0, q_1, q_2, q_3, q_4\}$, where q_i corresponds to strings w , where $\text{val}(w) \equiv_5 i$. q_0 is the start state, and $F = \{q_3\}$. The transition relation is defined as

$$\delta(q_i, \sigma) = q_j,$$

where $q_i, q_j \in Q$, $j = ((2i + \text{val}(\sigma)) \bmod 5)$, and $\sigma \in \{0, 1\}$.

(b) The DFA in Figure 1 recognizes language L_2 .

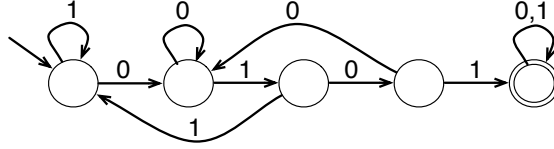


Figure 1: DFA M such that $L(M) = L_2$ in Problem 4b.

(c) The DFA in Figure 2 recognizes language L_3 . State i , for $i \in \{0, 1, 2, -1, -2\}$ corre-

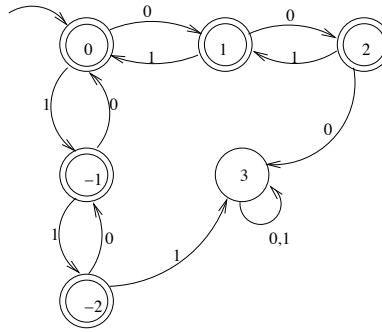


Figure 2: DFA M such that $L(M) = L_3$ in Problem 4c.

sponds to strings w in which $\#_0(w) - \#_1(w) = i$. State 3 corresponds to the strings w in which $|\#_0(w) - \#_1(w)| \geq 3$.

(d) Define the DFA $M = (K, \Sigma, \delta, s, A)$ where

- $K = \{q_{ijk} : 0 \leq i, j, k \leq 4\}$ (125 states), where q_{ijk} is the state that denotes that in the string w read so far, we have $\#_a(w) \equiv_5 i$, $\#_b(w) \equiv_5 j$, and $\#_c(w) \equiv_5 k$.
- $\Sigma = \{a, b, c\}$.
- $\delta(q_{ijk}, a) = q_{i'jk}$, where $i' = (i + 1) \bmod 5$
 $\delta(q_{ijk}, b) = q_{ij'k}$, where $j' = (j + 1) \bmod 5$
 $\delta(q_{ijk}, c) = q_{ijk'}$, where $k' = (k + 1) \bmod 5$.
- $s = q_{000}$.
- $A = \{q_{012}\}$.