

COMP 481: Automata, Formal Languages, and Computability  
 Spring 2008  
 Solutions to Homework Assignment #3

1. (a)  $L_a$  is not regular and hence has an infinite number of equivalence classes, where each string  $w \in \Sigma^*$  is in an equivalence class by itself.
- (b)  $L_b$  has an infinite number of equivalence classes, and they are

$$[a^i] = \{w \in \{a, b\}^* : \#_a(w) - \#_b(w) = i\},$$

for  $i \geq 0$ , and

$$[b^i] = \{w \in \{a, b\}^* : \#_b(w) - \#_a(w) = i\},$$

for  $i \geq 1$ .

- (c) The language is regular, and hence has a finite number of equivalence classes, which are:

- all strings that have  $bbb$  as a substring.
- all strings that do not have  $bbb$  as a substring and do not end with  $b$ .
- all strings that do not have  $bbb$  as a substring and the last symbol from the right is a  $b$  that is not preceded by a  $b$ .
- all strings that do not have  $bbb$  as a substring and the last two symbols from the right are  $bb$ .

It is easy to draw an FSM with four states, each of which corresponds to exactly one of these four equivalence classes.

2. In the first step, the following pairs are marked:  $(1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 6), (4, 6), (5, 6)$ .

In the second step, the following pairs become marked:  $(1, 6), (2, 6)$ .

In the third step, nothing gets marked, and we stop. Therefore, we have the following groups of equivalent states:  $p = \{1, 2\}$ ,  $q = \{3, 4, 5\}$ , and  $r = \{6\}$ . The minimized automaton is  $M = (Q, \Sigma, \delta, q_0, F)$ , where  $Q = \{p, q, r\}$ ,  $\Sigma = \{a, b\}$ ,  $\delta = \{(p, a, p), (p, b, q), (q, a, q), (q, b, r), (r, a, r), (r, b, r)\}$ ,  $q_0 = p$ , and  $F = \{q\}$ .

3. (a)  $(a \cup b(ab)^*(b \cup aa))(ba \cup (a \cup b)(ab)^*(b \cup aa))^*$
- (b)  $(a \cup b)a^* \cup (a \cup b)a^*b((a \cup ba \cup bb)a^*b^*)^*(a \cup ba \cup bb)a^*$
- (c)  $b^* \cup b^*a(ab + aab^*a)^*aab^* \cup b^*a(ab \cup aab^*a)^*b(b^* \cup b^*a(ab \cup aab^*a)^*aab^*)^*(bab^* \cup (bb \cup bab^*a)(ab \cup aab^*a)^*aab^*)$
4. (a) If there exists a string  $w$ , where  $|w| < n$ , such that  $w \in L(M)$ , then  $L(M) \neq \emptyset$ . Trivial. We now prove the other direction. Assume  $L(M) \neq \emptyset$ . Let  $w \in L(M)$  be the shortest string in  $L(M)$ . If  $|w| < n$ , then we are done. Otherwise,  $|w| \geq n$  and the Pumping

Lemma applies. In other words,  $w = xyz$ , where  $|xy| \leq n$ ,  $|y| \geq 1$ , and  $w' = xy^0z = xz \in L(M)$ . Clearly,  $|w'| < |w|$ . However, this contradicts the assumption that  $w$  is a shortest string in  $L(M)$ . Therefore, there must exist a string in  $L(M)$  whose length is less than  $n$ .

- (b) If  $M$  accepts a string  $w$ , where  $n \leq |w|$ , then by the Pumping Lemma,  $w = xyz$ , where  $|xy| \leq n$ ,  $|y| \geq 1$ , and  $xy^iz \in L$  for every  $i \geq 0$ . Therefore,  $L(M)$  is infinite.

For the other direction, assume  $L(M)$  is infinite. Therefore, there must exist at least one string whose length is  $\geq 2$  (the number of strings whose length is  $< n$  is  $\sum_{i=0}^n |\Sigma|^i$ , which is finite). Let  $w$  be a shortest such a string. If  $|w| < 2n$ , then we are done. Otherwise, assume  $|w| = p \geq 2n$ . By the Pumping Lemma,  $w = xyz$ , where  $|xy| \leq n$ ,  $|y| \geq 1$ , and  $xy^iz \in L$ . In particular,  $w' = xy^0z = xz \in L$ . Since  $|y| \leq n$ , then  $|xz| \geq n$ . Since  $w = xyz$  is a *shortest* string that satisfies  $|w| \geq 2n$ , then it must be that  $|xz| < 2n$ . Therefore,  $w' \in L(M)$  and  $n \leq |w'| < 2n$ . This completes the proof.