

COMP 481: Automata, Formal Languages, and Computability  
Spring 2009  
Solutions to Homework Assignment #8

1. I used  $R$  and  $D$  interchangeably to denote the set of recursive (decidable) languages, and use  $RE$  and  $SD$  interchangeably to denote the set of recursively enumerable (semi-decidable) languages.
2. I make use of the facts that  $A_{TM} = \{\langle M, w \rangle : M \text{ accepts } w\}$  is in  $SD$  but not in  $D$ , and that  $\overline{A_{TM}} = \{\langle M, w \rangle : M \text{ does not accept } w\}$  is not in  $SD$ .
3. I write  $\tau$  for a reduction function.

1. Let  $E$  be an enumerator for  $A$  and consider the enumerator  $E'$  which works as follows:

- Let  $x = a$
- Run  $E$ 
  - When  $E$  prints  $\langle M \rangle$ , run  $M$  on  $x$
  - If  $M$  rejects  $x$ , print  $x$ .
  - Assign to  $x$  the lexicographically next string in  $\Sigma^*$ .

Let  $\langle M_1 \rangle, \langle M_2 \rangle, \dots$  be the TMs enumerated by  $E$ . Notice two important properties of  $E'$ :

- (a)  $E'$  prints the  $i^{\text{th}}$  string in the lexicographical order of  $\Sigma^*$  if and only if  $M_i$  rejects the string. In other words, the language enumerated by  $E'$  is different from  $L(M_i)$  for every  $M_i$  enumerated by  $E$ .
- (b)  $E'$  prints strings in lexicographical order.

Therefore  $E'$  is a lexicographical enumerator of a recursive language  $L$  of which none of the TMs enumerated by  $E$  is a decider.

2. We show a reduction  $f$  from  $A_{TM}$  to  $T$ .  $f(\langle M, w \rangle) = \langle M' \rangle$ , where  $M'$  on input  $x$  works as follows:

- If  $x = ab$ , accept.
- else, run  $M$  on  $w$ . If  $M$  accepts  $w$ , accept.

If  $M$  accepts  $w$ , then  $L(M') = \Sigma^*$ . Hence,  $\langle M' \rangle \in T$ .

If  $M$  does not accept  $w$ , then  $L(M') = \{ab\}$ . Hence,  $\langle M' \rangle \notin T$ .

Since  $A_{TM} \notin R$ , it follows that  $T \notin R$ .

3. The proof is in your textbook!

4. If  $A \leq_m A_{TM}$  then  $A \in RE$ , since  $A_{TM} \in RE$  (follows from the theorem we learned in class).

Now, assume that  $A \in RE$ . We show that  $A \leq_m A_{TM}$ . We need to show a function  $f : \Sigma^* \rightarrow \Sigma^*$ , where  $w \in A$  if and only if  $f(w) \in A_{TM}$ . Let  $M$  be a TM that semidecides  $A$ , and let  $f$  be the function  $f(w) = \langle M, w \rangle$ .

If  $w \in A$ , then  $M$  accepts  $w$  (since  $M$  is a semidecider of  $A$ ). Hence,  $\langle M, w \rangle \in A_{TM}$ .

If  $w \notin A$ , then  $M$  does not accept  $w$ . Hence,  $\langle M, w \rangle \notin A_{TM}$ . Clearly,  $f$  is computable. Therefore,  $f$  is a mapping reduction from  $A$  to  $A_{TM}$ .

5. Since  $0^*1^* \in R$ , then  $A \leq_m 0^*1^*$  implies that  $A \in R$ .

Now, assume  $A \in R$ . We show that  $A \leq_m 0^*1^*$ . Let  $M$  be a decider for  $A$ , and let  $f$  be the function computed by the following TM  $M'$ .  $M'$  on input  $w$ : If  $M$  accepts  $w$ , then  $f(w) = 01$ ; if  $M$  rejects  $w$ , then  $f(w) = 10$ . Since  $M$  is a decider,  $M'$  always halts, and  $f$  is computable.

If  $w \in A$ , then  $f(w) = 01 \in 0^*1^*$ . And if  $w \notin A$ , then  $f(w) = 10 \notin 0^*1^*$ . Therefore,  $f$  is a mapping reduction from  $A$  to  $0^*1^*$ .

6. Let  $M_{3x+1}$  be the following TM.  $M_{3x+1}$  on input  $x$ :

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While ( $x \neq 1$ )
   $x \leftarrow f(x)$ .
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Let  $M$  be the TM that works as follows on its input  $x$ :

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For  $n = 1$  to  $\infty$ 
  Run  $H$  on the pair  $\langle M_{3x+1}, n \rangle$ .
  If  $H$  rejects, halt and accept.
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In other words,  $M$  is a TM that halts and accepts only if there is a number that doesn't give a sequence that ends in 1. Notice that the assumption that  $H$  exists was crucial in designing  $M$ . Otherwise, we couldn't have written "If  $H$  rejects, halt and accept."

We are not done yet. The question is: Does  $M$  halt (on any arbitrary input, since the input to  $M$  really doesn't matter)? If it does, then there is a number that doesn't give a sequence that ends in 1. If it doesn't halt, then there doesn't exist such a number; i.e., all numbers give a sequence that ends in 1. But this is easy to check. Assuming that  $H$  exists, run  $H$  on  $\langle M, a \rangle$  ( $a$  is just an arbitrary input; we could have used any string). If  $H$  accepts, then the answer to the  $3x + 1$  problem is NO (not all positive starting points end up a 1); if  $H$  rejects, then the answer is YES. Hence, we have a decider for the  $3x + 1$  problem.

7. We reduce  $\overline{A_{TM}}$  to both  $S$  and  $\overline{S}$ . To get a reduction to  $S$ , let  $f$  be the function, where  $f(\langle M, w \rangle) = \langle M' \rangle$  where  $M'$  works as follows on input  $x$ :  
—Obtain own description  $\langle M' \rangle$  (via the Recursion Theorem)

—If  $x = \langle M' \rangle$ , accept  
 —else, run  $M$  on  $w$ .

Clearly, if  $M$  doesn't accept  $w$ , then  $L(M') = \{\langle M' \rangle\}$ . If  $M$  accepts  $w$ ,  $L(M') = \Sigma^*$ . Therefore,  $\langle M, w \rangle \in \overline{A_{TM}}$  iff  $\langle M' \rangle \in S$ . Hence,  $S \notin RE$ .

To get a reduction to  $\overline{S}$ , let  $f$  be the function, where  $f(\langle M, w \rangle) = \langle M' \rangle$  where  $M'$  works as follows on input  $x$ :

—Obtain own description  $\langle M' \rangle$  (via the Recursion Theorem)  
 —run  $M$  on  $w$ . If  $M$  accepts and  $w = \langle M' \rangle$ , accept; else, reject.

Clearly, if  $M$  doesn't accept  $w$ , then  $L(M') = \emptyset$ . If  $M$  accepts  $w$ ,  $L(M') = \{\langle M' \rangle\}$ . Therefore,  $\langle M, w \rangle \in \overline{A_{TM}}$  iff  $\langle M' \rangle \in \overline{S}$ . Hence,  $\overline{S} \notin RE$ .

8. (a)  $C = \{L \in RE : \exists x, |x| \equiv_5 1, \text{ and } x \in L\}$ . Clearly,  $C \subseteq RE$ . We also have that  $\emptyset \in RE$  but  $\emptyset \notin C$ . Further,  $\Sigma^* \in C$ . Hence,  $C$  is a nontrivial subset of  $RE$ . By Rice's theorem, it follows that  $L_C = \{\langle M \rangle : L(M) \in C\}$  is not in  $R$ .  $L_C = \{\langle M \rangle : \exists x, |x| \equiv_5 1, \text{ and } x \in L(M)\} = L_1$ . Therefore,  $L_1 \notin R$ .
- (b)  $C = \{L \in RE : L \text{ contains all palindromes}\}$ . You can easily prove that  $C$  is a nontrivial subset of  $RE$ , and conclude that  $L_2$  is not in  $R$ .
- (c)  $C = \{L \in RE : L \text{ does not contain any string } w \text{ such that } 001 \text{ is a prefix of } w\}$ . Proving that  $C$  is a nontrivial subset of  $RE$  is trivial in this case too. It follows that  $L_3$  is not in  $R$ .

9. (a)  $S1 \rightarrow 111S, SS \rightarrow 11111$ .

(b)  $111 \rightarrow \varepsilon, SS \rightarrow 1, S1S \rightarrow 11, S11S \rightarrow 111$ .

(c)  $Sa \rightarrow aaS, Sb \rightarrow bbS, SS \rightarrow \varepsilon$ .

(d)  $Sa \rightarrow aLA \quad Sb \rightarrow bLB$   
 $aS \rightarrow aR\# \quad bS \rightarrow bR\#$   
 $La \rightarrow aLA \quad Lb \rightarrow bLB$   
 $X\alpha \rightarrow \alpha X \quad \text{for every } X \in \{A, B\} \text{ and every } \alpha \in \{a, b, R\}$   
 $A\# \rightarrow a\# \quad B\# \rightarrow b\#$   
 $LR \rightarrow \varepsilon$   
 $\# \rightarrow \varepsilon$

10. (a)  $S \rightarrow S_1XT$   
 $S_1 \rightarrow aS_1a|bS_1b|\#TY$   
 $T \rightarrow aT|bT|\varepsilon$   
 $Y \rightarrow YA$   
 $Aaa \rightarrow aAa$   
 $Abb \rightarrow bAb$   
 $Aba \rightarrow aAb$   
 $Aab \rightarrow bAa$

$AaX \rightarrow Xa$   
 $AbX \rightarrow Xb$   
 $YX \rightarrow \varepsilon.$

(b)  $S \rightarrow ABSC|ABS|T$

$T \rightarrow AT|\varepsilon$   
 $AB \rightarrow BA$   
 $BA \rightarrow AB$   
 $BC \rightarrow CB$   
 $CB \rightarrow BC$   
 $AC \rightarrow CA$   
 $CA \rightarrow AC$   
 $A \rightarrow a$   
 $B \rightarrow b$   
 $C \rightarrow c.$

(c)  $S \rightarrow ABSXY, S \rightarrow ABcXY$

$YX \rightarrow XY, BA \rightarrow AB$   
 $Bc \rightarrow bc, Bb \rightarrow bb$   
 $Ab \rightarrow ab, Aa \rightarrow aa$   
 $cX \rightarrow ca, aX \rightarrow aa$   
 $aY \rightarrow ab, bY \rightarrow bb$

(d)  $S \rightarrow aBSccc|\varepsilon$

$Ba \rightarrow aB$   
 $Bc \rightarrow bbc$   
 $Bb \rightarrow bbb.$

(e)  $S \rightarrow aBSd|T$

$T \rightarrow bTc|\varepsilon$   
 $Ba \rightarrow aB$   
 $Bb \rightarrow bb$   
 $Bd \rightarrow bd.$