

Comp 411 Notes - Mon, Feb 7 2005

TYPE PRESERVATION

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In today's lecture, we prove a theorem of *type preservation*. We begin by reviewing substitutivity, and then identify and prove two lemmas on type preservation in β reductions. These two lemmas complete the proof of the theorem.

1 Substitutivity Review

Substitutivity shows that IF you have progress in reduction, the original and reduced term have the same type.

Lemma 1.1 *Substitutivity Lemma:*

$$\frac{\Gamma, x : t_2 \vdash e_1 : t_1 \quad \Gamma \vdash e_2 : t_2}{\Gamma \vdash e_1[x := e_2] : t}$$

2 Theorem: Type Preservation

Theorem 2.1 *Type Preservation*

$$\Gamma \vdash e_1 : t_1 \wedge e_1 \rightarrow e_2 \Rightarrow \Gamma \vdash e_2 : t_1$$

To prove this, we must first revisit some definitions:

Definition 2.2 *Notion of Reduction*

$$(\lambda x. e_1) e_2 \rightarrow_{\beta} e_1[x := e_2]$$

Definition 2.3 *Context*

$$C ::= [] | eC | Ce | \lambda x. C$$

Definition 2.4 *Compatible Extension of β (Reduction)*

$$\frac{e_1 \rightarrow e_2}{C[e_1] \rightarrow C[e_2]}$$

We need to establish a few more points to prove our theorem. In class, we began with an **incorrect** lemma stated thus:

Lemma 2.5 *Compatible Extension of β Preserves Types (incorrect)*

$$\Gamma C[e_1] : t \Rightarrow \Gamma \vdash C[e_2] : t$$

This is clearly wrong because it states that any expression of the form $C[e]$ has the same type as any other expression of that form.

This lemma is almost correct when we add in the following two constraints:

$$\Gamma \vdash e_1 : t_2$$

$$\Gamma \vdash e_2 : t_2$$

The correct formulation of this lemma is actually:

Lemma 2.6 *Compatible Extension of β Preserves Types (correct)*

$$\Gamma \vdash C[e_1] : t \rightarrow \exists \Gamma', t'. \Gamma' \vdash e_1 : t' \wedge \forall e_2. \Gamma' \vdash e_2 : t' \Rightarrow \Gamma \vdash C[e_2] : t$$

We need to prove one additional lemma:

Lemma 2.7 *β Preserves Typing*

$$\Gamma \vdash e_1 : t \wedge e_1 \rightarrow_{\beta} e_2 \Rightarrow \Gamma \vdash e_2 : t$$

To prove this lemma, first observe that only lambda expressions can reduce. Therefore, we can rephrase the lemma as:

$$\Gamma \vdash e_1 : t \wedge e_1 \rightarrow_{\beta} e_2 \Rightarrow \Gamma \vdash e_2 : t$$

Also notice that the outcome of this lemma is the same as the outcome of the substitutivity lemma. Therefore to prove that this lemma is true, all that is required is to show that its starting assumptions imply the starting assumptions of the substitutivity lemma.

We will do this using a *Reverse Typing Derivation*. In other words, we will start from an expression and derive *backwards*.

Proof:

Step 1 We start with what we are given.

$$\Gamma \vdash (\lambda x. e_1) e_2 : t$$

Step 2 We take one (derivation) step backwards.

$$\frac{\Gamma \vdash \lambda x.e_1 : t_1 \rightarrow t \quad \Gamma \vdash e_2 : t_1}{\Gamma \vdash (\lambda x.e_1)e_2 : t}$$

Step 3 One more step takes us to a point where we have the same assumptions as the start of the substitutivity lemma.

$$\frac{\frac{\Gamma, x : t_1 \vdash e_1 : t}{\Gamma \vdash \lambda x.e_1 : t_1 \rightarrow t} \quad \Gamma \vdash e_2 : t_1}{\Gamma \vdash (\lambda x.e_1)e_2 : t}$$

The two lemmas we have defined completes the proof of the theorem.

3 Other

It is worth noting that this proof also can be applied to β -value reductions.

We can also conceive of the small step semantics:

$$E[e_1] \rightarrow E[e_2]$$

where

$$E ::= [] \mid vE \mid Ee$$