Comp 411 Notes - 21 Jan 2005

EQUIVALANCE OF BIG-STEP AND SMALL-STEP SEMANTICS

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1 Important Points from Previous Lecture

1. As a reminder, expressions are

$$\begin{aligned} e \in E ::= \; & T|F|if\ e\ then\ e\ else\ e| \\ & 0|e+|e-|iszero\ e \end{aligned}$$

   - T and F as boolean constructors, and if as the elimination expression.
   - Likewise 0 and $e+$ are constructors, and $iszero$ is the elimination expression.
   - $e-$ is not quite either. Being partial doesn't help.

2. Values are:

$$v \in V ::= T|F|0|v+$$

   - Note that $v-$ is not in the set of values, as it is never in the result of any of the rules in the big-step semantics.
   - If there are too many elements in your set of values, idempotence of values can break.
   - Idempotence is: $v \hookrightarrow v' \Rightarrow v = v'$ or $\forall v \in V v \hookrightarrow v$
   - Namely, $v-$ would not always evaluate to $v-$: $0+- \hookrightarrow 0$

2 Induction

- Induction is something defined on the natural numbers: $n ::= 0 \| n+$
  - Prove a theorom for 0.
  - Assume that it is true for n
  - Prove it is true for $n+$.
- Intuition for the assumption: there is no other way to construct values in this set.
- Reality of the assumption: an axiom in constructive logic. Possibly derivable from the principle of the excluded middle in classical logic.
- Induction over expressions. Expressions are not natural numbers.
  - One hand: you can assume that a theorem is true for all the sub-components of e as an analog to assuming it is true for n.
  - Other hand: create set S0, empty. S1 is the set of trees of length one, etc.
    Then we can index families of sets by the natural numbers, and we can do proofs by inductionover the indexes of this set.
  - Gripping hand: we are doing induction over the height of the derivation tree.
- Strong induction is: assume true for n and all things smaller than n. It is not necessary if we construct our sets as follows.

$$S_0 = \{\}$$
$$S_{n+1} = S_n \cup ExtendR(Sn)$$

3 Proof of equivalance of big step and small step semantics

- The proof is $e \hookrightarrow v \Leftrightarrow e \mapsto^* v$
- But can be split into two parts.

  1. $e \hookrightarrow v \Rightarrow e \mapsto^* v$
  2. $e \mapsto^* v \Rightarrow e \hookrightarrow v$

- Proving 1
- $\dfrac{}{T \hookrightarrow T}$ has no corresponding rule, but the kleene star is the same as saying

$$\exists k \text{ such that } e \mapsto^k v$$

  and so for R_t $k = 0$.

- R_f, R_0 are the same.
- So onto if.

$$\frac{e_1 \hookrightarrow T \quad e_2 \hookrightarrow v}{if\ e_1\ then\ e_2\ else\ e_3 \hookrightarrow v}$$

  - We will construct a proof by induction on height of the tree represented by $e \hookrightarrow v$, and case analysis on last rule in the tree.
  - What we want to do for $e \hookrightarrow v$ is to construct the corresponding sequence of tree in small-step semantics. $if\ e_1\ then e_2 else e_3 \mapsto^* v$
  - There are two rules we might have to match, R_{ifT} and R_{ifF}. Without loss of generality, assume $e_1 \mapsto^* T$

- Our inductive hypotheses:

  1. $e_1 \mapsto^i T$
  2. $e_2 \mapsto^j v$

- Small-step rules Rules for if look either like a-type

$$\frac{e_1 \hookrightarrow e_1'}{if\ e_1\ then\ e_2\ else\ e_3 \mapsto if\ e_1'\ then\ e_2\ else\ e_3}$$

- or b-type

$$\frac{}{if\ T\ then\ e_2\ else\ e_3 \mapsto e_2}$$

- Note the later is an elimination rule.
- The intution is that the sequences of trees that yield $if\ e_1\ then\ e_2\ else\ e_3 \mapsto^* v$ will look like

  1. i trees where the final rule is an a-type rule, that corresond to the i trees that yield $e_1 \mapsto^i T$
  2. One tree of the b-form
  3. And then j trees that are exactly the j trees that yield $e_2 \mapsto^j v$

- Thus $if\ e_1\ then\ e_2\ else\ e_3 \mapsto^{i+j+k} v$
- The technicality here is that we are not exactly using our first inductive hypothesis. That hypothesis takes us from e_1 to T, not from $if\ e1...$ to $if\ T...$

  1. The i trees in the hypothesis are of height $n_1...n_i$
  2. The i trees in the usage are of height $n_1 + 1...n_1 + 1$, because we have the extra a-type rule at the bottom.
  3. Note there are still i trees, there is just that extra rule on the bottom.
  4. Thus we need a little lemma saying $\forall(e_1 \mapsto^i T)\exists(if\ e_1.. \mapsto^i if\ T....)$
  5. This lemma will be proved with induction over the length i of the chain of derivation trees.