Comp 411 Notes - 28 Jan 2005

SUBSTITUTION

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1 Substitution Defined

Given:

$$e \in \mathbb{E} \quad ::= \quad x|\lambda x.e|e\ e$$

where $x \in \mathbb{X}$, a countably infinite set.

We define substitution as a 3-ary function $\mathbb{E} \times \mathbb{X} \times \mathbb{E} \rightarrow \mathbb{E}$; syntactically, it takes the form $e_1[x := e_2] = e_3$. The valid values for e_3 are described by induction on e_1 as follows:

$$y[x := e_2] = \begin{cases} e_2 & \text{if } x = y \\ y & \text{if } x \neq y \end{cases} \tag{1}$$

$$(\lambda y.e_4)[x := e_2] = \lambda z.(e_4[y := z][x := e_2]) \text{ where } z \text{ is "fresh"} \tag{2}$$

$$e_4\ e_5[x := e_2] = e_4[x := e2]\ e_5[x := e2] \tag{3}$$

Question: does "fresh," as used above, mean "witty?" Answer: no. "Fresh" means that $z \notin FV(e_4) \cup FV(e_2) \cup \{y\}$. ($FV(e)$ is the set of all free variables in e.)

2 Properties of Substition

Notation: $\exists! e.P(e)$ means that exactly one e satisfies $P(e)$; more formally, it is equivalent to $\exists e.(P(e) \wedge (\exists e'.P(e') \Rightarrow e = e'))$.

Similarly, if we define an equivalence $=_x$, we can say $\exists!_x e.P(e)$ meaning that exactly one equivalence class satisfies $P(e)$; more formally, it is equivalent to $\exists e.(P(e) \wedge (\exists e'.P(e') \Rightarrow e =_x e'))$.

First we consider whether substitution is really a total function—that is,

$$\forall e_1, e_2, x, \exists! e'.e' = e_1[x := e_2]$$

Since our notion of "fresh" is nondeterministic, it appears that substitution must be nondeterministic as well. (We could define "fresh" in a functional manner, but we choose not to.) For example, λ`a.b a`[`b` := `newb`] = λ`c.newb c`; but also λ`a.b a`[`b` := `newb`] = λ`j.newb j`. So the truth of this property rests on the meaning of =. In general, does $\lambda x.x = \lambda y.y$? While the semantic interpretation of the two is equivalent, mathematical equality is defined in terms of syntax, and there are obviously values for x and y such that the above statements are syntactically different.

Since the previous property did not hold, we accept a weaker assertion:

$$\forall e_1, e_2, x, \exists e'.e' = e_1[x := e_2]$$

But since (informally) substitution really does give the "same" result for some specific input, even if the variable names used are different, we will define a new notion of equivalence: $=_\alpha \subseteq \mathbb{E} \times \mathbb{E}$. We define $e_1 =_\alpha e_2$ inductively on e_1:

$$\frac{}{x =_\alpha x} \qquad\qquad \frac{e_3 =_\alpha e_5 \wedge e_4 =_\alpha e_6}{e_3\ e_4 =_\alpha e_5\ e_6}$$

$$\frac{z \notin FV(e_3) \cup FV(e_4) \wedge e_3[x_1 := z] =_\alpha e_4[x_1 := z]}{\lambda x_1.e_3 =_\alpha \lambda x_2.e_4}$$

Having defined $=_\alpha$, we can restate our initial assertion as follows:

$$\forall e_1, e_2, x, \exists!_\alpha e'.e' =_\alpha e_1[x := e_2]$$

3 Homework

Due with next problem set: Prove

$$\forall e_1, e_2, x, \exists e'.e' = e_1[x := e_2]$$

and

$$\forall e_1, e_2, x, \exists!_\alpha e'.e' =_\alpha e_1[x := e_2]$$