

Comp 411 Notes - Fri, Feb 23, 2005

Subtyping (Continued)

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This lecture defines and discusses the rules of the subtyping relation. It first presents motivating examples for the rules, and then presents them in their general form.

1 Motivating Examples

Intuitively, a value in the subtype could replace a value of the supertype. In the subtyping relation, $<$, we thus could have:

1. $\{x : Nat, y : Nat\} <: \{x : Nat\}$.
 2. $\{x : Nat, y : Nat\} <: \{y : Nat, x : Nat\}$.
 3. $\{x : Nat\} <: \{x : Nat\}$.
 4. (a) Contravariant subtyping:
 $\{x : Nat\} \rightarrow bool <: \{x : Nat, y : Nat\} \rightarrow bool$
Motivation for contravariance:
 - Typable Example:
 $\lambda a : (\{x : Nat\} \rightarrow A). \lambda b : \{x : Nat, y : Nat\}. (a\{x = b.x\})$
 - Non-Typable Example:
 $\lambda a : (\{x : Nat, y : Nat\} \rightarrow A). \lambda b : \{x : Nat\}. (a\{x = b.x, y = ???\})$. (No input-dependent value can be used to fill for field y)
 - (b) Covariant subtyping:
 $A \rightarrow \{x : Nat, y : Nat\} <: A \rightarrow \{x : Nat\}$.
5. $\{x : Nat, y : Nat, z : Nat\} <: \{x : Nat, y : Nat\}$: All record types that have the fields of the subsets of the set of all fields (can u read that?!) (i.e., the set $\{x, y, z\}$), e.g., $\{y : Nat, z : Nat\}$ ($\{\}$ is a special case).
 6. $\{x : \{y : Nat, z : Nat\}\} <: \{y : \{y : Nat\}\}$.

2 The Subtyping Rules

More generally, we have:

0. Transitivity:
$$\frac{A <: B \quad B <: C}{A <: C}$$

1. Width: $\frac{}{\{x_i : t_i\}^{i \in I \oplus J} <: \{x_i : t_i\}^{i \in I}}$
2. Permutation: $\frac{I \text{ is a permutation of } J}{\{x_i : t_i\}^{i \in I} <: \{x_i : t_i\}^{i \in J}}$
3. Reflexivity: $\frac{}{A <: A}$
4. Arrows:
 - (a) Contravariance: $\frac{B_2 <: B_1}{B_1 \rightarrow A <: B_2 \rightarrow A}$
 - (b) Covariance: $\frac{B_1 <: B_2}{A \rightarrow B_1 <: A \rightarrow B_2}$
 - (c) Both: $\frac{A_1 <: A_2 \quad B_1 <: B_2}{B_2 \rightarrow A_1 <: B_1 \rightarrow A_2}$
5. Top: $\frac{}{A <: Top}$
6. Depth: $\frac{\{s_i <: t_i\}}{\{x_i : s_i\}^{i \in I} <: \{x_i : t_i\}^{i \in I}}$ (*Curly brackets in the antecedent are set brackets, not record brackets*)

Yet, all of these rules are *not* enough. So far, subtyping would be a relation on its own, having nothing to do with typing, unless the following important typing rule is added:

$$\frac{\Gamma \vdash e : t_1 \quad t_1 <: t_2}{\Gamma \vdash e : t_2} \text{ (Subsumption)}$$

The rule, in its given format, however, is problematic. It can be applied many (an infinite) number of times. In next lecture it would be wired into the application rule, and that would be provably enough (have the same observable behavior as having it as an independent rule).