COMP 409, Lecture 13

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1 Deductions

A **Deductive System** is a set $\Gamma \subseteq (Form)^+$ of inference rules. An *inference rule* $\varphi_1 ... \varphi_k \in \Gamma$ (usually written as $\frac{\varphi_1,\ldots,\varphi_{k-1}}{\varphi_k}$) means that φ_k follows from $\varphi_1, ..., \varphi_{k-1}$. A special case of this is $\frac{}{\varphi}$, which is called an *axiom*.

Definition 1 *A deductive system* Γ *is sound if whenever* $\vdash_\Gamma \varphi$ *we have* $\models \varphi$

Soundness of a deductive system is an absolute must. The simplest system with this property is $\Gamma = \emptyset$.

Definition 2 *A deductive system is complete if whenever* $\models \varphi$ *we have* $\vdash_\Gamma \varphi$

Completeness is desirable but not always attainable. The simplest system with this property is $\Gamma = \Sigma^*$. Sound but incomplete systems could be interesting, but complete but unsound systems are not interesting.

This does not necessarily give us interesting deductive systems. For example, $\Gamma = \{\frac{\varphi_1,\ldots,\varphi_k}{\varphi} | \{\varphi_1, ..., \varphi_k\} \models \varphi\}$ is sound and complete, but uninteresting. What we really want in a deductive system is soundness, completeness, and Γ is polynomially tractable.

We define $Deductions(\Gamma)$ to be all of the deductions relative to Γ. We desire $Deductions(\Gamma)$ to be in $PTIME$ (that is, all deductions can be verified in $PTIME$):

Definition 3 *A deductive system* Γ *is* tractable *if* $Deductions(\Gamma) \in PTIME$.

Definition 4 *An inference rule* $\frac{\varphi_1,\ldots,\varphi_{k-1}}{\varphi_k}$ *is sound if* $\models \varphi_k$ *hold, when ever* $\models \varphi_1, \ldots \models \varphi_{k_1}$ *hold.*

A special case is when $varphi_1, \varphi_2, \ldots \varphi_{k-1} \models \varphi_k$ holds. Note that an axiom is sound when it is valid.

Theorem 1 Γ *is sound if all its rules are sound.*
Proof *By induction on length of deductions.*

- *Base: If* $< \varphi >$ *is a* Γ *deduction, then* $< \varphi > \in \Gamma$*, so* $\models \varphi$.

- *Induction: If φ_1,...,φ_k is a Γ-deduction. By induction hypothesis, $\models\varphi_i$, for 1$\leq$i$\leq$k-1. There are i_1,...,i_k¡k such that $\frac{\varphi_{j_1}\varphi_{j_2}\cdots\varphi_{j_{k-1}}}{\varphi_j} \in \Gamma$. Therefore, $\models\varphi_k$.*

Now suppose somebody gives you $\varphi_1, ..., \varphi_n$ and tells you to check whether it is a deduction or not i.e. he asks you to check a proof. An answer to this could be 'For each φ_i find $\varphi_{i_1}\varphi_{i_2}...\varphi_{i_k}$ such that $i_1, i_2, ... < i$ and $\frac{\varphi_{i_1},...,\varphi_{i_k}}{\varphi_i} \in \Gamma$. If we succeed for all φ_i, the given sequence is a deduction.' But we have to make sure that each time we check for φ_i we do it in polynomial time, i.e., we have to ensure that each time the possible number of choices of antecedents($(i-1)^k$) should be bounded(which occurs when k is bounded) and we should be able to check membership in polynomial time. Now we state these conditions, under which a deductive system is polynomially tractable, more formally. We define arity of a deductive system as follows.

Definition 5 *Γ is l-ary if every rule in Γ has at most l antecedents.*

Theorem 2 *If*

1. $\Gamma \in PTIME$
2. $\Gamma \subseteq (Form)^k$ *for some* $k > 0$

then Γ is polynomially tractable.

2 Sound and Complete Deductive System for Propositional Logic

- Hilbert's System:
  Hilbert discovered a very elagant deductive system that is both sound and complete. His system uses only the connectives $\{\neg, \rightarrow\}$, which we know are an adequate basis. It has also been shown that each of the 3 axioms given below are necessary for the system to be sound and complete. Hilbert's System consists of the following.
  - Three axioms:
    * A1: $(p \rightarrow (q \rightarrow p))$
    * A2: $((p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r)))$
    * A3: $((\neg p \rightarrow q) \rightarrow ((\neg p \rightarrow \neg q) \rightarrow p))$
  - Inference Rules:
    * Modus ponens: $\frac{p, p\rightarrow q}{q}$
    * Substitution: $\frac{\varphi}{\varphi[\theta]}$; θ is a substitution

Now we check whether Hilbert's System is polynomially feasible or not. Clearly arity of Hilbert's System is two since the maximum number of antecedents in an inference rule in the system is two. So the first condition of polynomial tractability holds. To find whether a deduction can be checked in polynomial time, we have to find whether it is an axiom or modus ponens or substitution. All these can be checked in polynomial time by pattern matching if we think of axioms and rules as patterns and can parse the formulae(to find substitutions). So, Hilbert's rules and axioms are in polynomial time. Hence the corollary, Hilbert's System is polynomially tractabl.:

- φ, $\psi \in$Form
- Matching problem: Is there a substitution θ such that $\varphi[\theta]=\psi$.
- Matching$\in$PTIME

Theorem 3 *Hilbert's system is sound and complete.*

- Meredith's System, 1953):
  Logicians are always interested in the minimum number of axioms. So, Meredith found that he could combine the power of all three of Hilbert's axioms into the single axiom that follows: Axiom: $((((p \to q) \to (\neg r \to \neg s)) \to r) \to u) \to ((u \to p) \to (s \to p))$
  We retain the Inference Rules: modus ponens, substitution