Comp 409 Lecture 17

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Last time we proved completeness theorem for resolution. We showed that a a finite set of clauses isunsatisfiable iff it is refutable. Can we extend the theorem to infinite sets of clauses? The previous proof depended on the finiteness of *Prop*.

1 Compactness

Suppose $Prop$ is infinite, then there are infinitely many formulas, and we may potentially have problems refuting an infinite unsatisfiable set of formulas. In contrast, if $Prop = p_1, \ldots, p_n$, then for every $\varphi \in Form$, $\varphi : \{0,1\}^n \mapsto \{0,1\}$. There are only 2^{2^n} different Boolean formulas. So, if $Prop$ is finite, we only have finitely many formulas up to equivalence, and, therefore, a set of formulas is unsatisfiable iff some finite subset is unsatisfiable. We would like the same to hold if $Prop$ is infinite.

When we proved the completeness of resolution, we show that we can apply the operator R_p to "resolve away" proposition after propositions. When $Prop$ has n propositions, this guarantees that after n steps we end up wither with the empty clause or with the empty set of clauses. This argument does not hold when $Prop$ is infinite. Because resolution always has two antecedants, every refutation of a set H of formulas will consider only a finite subset of H. A priori, it is not clear that this is enough to show that H is unsatisfiable.

Definition *H is finitely satisfiable ($FSAT$) if every finite subset of H is satisfiable.*

The Compactness Theorem *$H \in SAT$ iff $H \in FSAT$.*

Proof The left-to-right direction of the biconditional is trivial. Clearly, if H is satisfiable by some truth assignment then every finite subset is satisfiable by the same truth assignment. Now suppose that $H \in FSAT$.

Contrapositive form of Compactness Theorem : A set Φ is unsatisfiable iff it has an unsatisfiable finite subset.

Theorem *If Φ is unsatisfiable, then Φ is refutable.*
Proof By Compactness theorem (contrapositive form), Φ has finite unsatisfiable subset Φ'. The claim follows by completeness of resolution.

We now proceed to prove the hard direction of the Compactness Theorem. Suppose we have a truth assignment $\tau : Prop \mapsto \{0, 1\}$.

Definition $Th(\tau) = \{\varphi | \tau \models \varphi\}$. *("$Th(\tau)$" is read as "the theory of τ").*

Lemma *If $\tau \models H$, then $H \subseteq Th(\tau)$.*

Definition *A set $H \subseteq Form$ of formulas is **exhaustive** iff for all $\varphi \in Form$, $\varphi \in H$ or $\neg\varphi \in H$.*

Take an arbitrary formula φ. Either $\tau \models \varphi$ or $\tau \not\models \varphi$. Therefore, either $\tau \models \varphi$ or $\tau \models \neg\varphi$, i.e., either $\varphi \in Th(\tau)$ or $\neg\varphi \in Th(\tau)$. We proved:

Lemma *$Th(\tau)$ is exhaustive.*

It is also clear that:

Lemma *$Th(\tau)$ is satisfiable, and therefore also finitely satisfiable (FSAT).*

We now prove an important property of exhaustive sets:

Lemma *If H is $FSAT$ and exhaustive, then H is satisfiable.*

Proof: Take $p \in Prop$. Either $p \in H$ or $\neg p \in H$. Define $\tau(p) = 1$ if $p \in H$, $\tau(p) = 0$ if $\neg p \in H$. We show that $\tau \models H$.

Claim: $\varphi \in H$ iff $\tau \models \varphi$.

Proof 1 (By Structural Induction)

Basis: We know that $p \in H$ iff $\tau(p) = 1$ iff $\tau \models p$.

Induction:

- φ is $\neg\psi$. Then $\varphi \in H$ iff $\psi \notin H$. The right-to-left part of the biconditional holds by the exhaustiveness of H. The left-to-right holds part by finite satisfiability of H (otherwise, $\psi, \neg\psi \in H$ and H is not $FSAT$). Then, by the Induction Hypothesis, $\varphi \in H$ iff $\tau \not\models \psi$ iff $\tau \models \varphi$.

- φ is $\theta \wedge \psi$. Then $\varphi \in H$ iff $\theta \in H$ and $\psi \in H$.

  $\leftarrow$

  Suppose $\theta \wedge \psi \in H$, but one of θ, ψ isn't. $WLOG$, we can assume that $\theta \notin H$. Then, by exhaustiveness of H, $\neg\theta \in H$. But since H is $FSAT$ this is impossible.

  $\rightarrow$

  Suppose $\theta \in H, \psi \in H, \theta \wedge \psi \notin H$. Then $\neg(\theta \wedge \psi) \in H$, by exhaustiveness of H. Then $\theta, \psi, \neg(\theta \wedge \psi)$ is not $FSAT$ (impossible).

Then by the inductive hypothesis, $\varphi \in H$ iff $\theta \in H$ and $\psi \in H$ iff $\tau \models \theta$ and $\tau \vdash \psi$ iff $\tau \models \varphi$. $\square$

(Notice that never used the full power of $FSAT$ in the proof of this lemma, as we only considered sets with two or three formulas.)

Proof 2 (By Contradiction)

Assume $\varphi \in H$, but $\tau(\varphi) = 0$. (If $\varphi \notin H$ but $\tau(varphi) = 1$, then $(\neg\varphi) \in H$ and $\tau((\neg\varphi) = 0$.) Let $M = \{\varphi, l_1, l_2, ...l_k\} \subseteq H$, where $AP(\varphi) = \{p_1, p_2, ..., p_k\}$ and

$$l_i = \begin{cases} \mathrm{p}_i & if p_i \in H \\ (\neg p_i) & if p_i \notin H \end{cases}$$

Claim: M is unsatisfiable, contradicting the finite satisfiability of H.
Proof: Let $\tau' : AP(\varphi) \to \{0.1\}$. Suppose $\tau' \models M$. Clearly, $\tau|_{AP(\varphi)} = \tau'$, so, Using Relevence Lemma, we get that $\tau(\varphi) = 1$, a contradiction. It follows that M is unsatisfiable.

So, now, suppose that H is $FSAT$, we need to find $\tau : Prop \mapsto \{0, 1\}$ s.t. $\tau \models H$. We know that:

1. $H \subseteq Th(\tau)$
2. $Th(\tau)$ is exhaustive
3. $FSAT + exhaustiveness \Rightarrow satisfiability$

Strategy: Find $H' \geq H$ such that H' is FSAT and exhaustive.

Lemma: If H is $FSAT$ then $H \subseteq H'$ where H' is $FSAT$ and exhaustive.

(Proviso: We are only considering countably large sets of propositions, and, therefore, countably large sets of formulas).

Proof: Take $\alpha_1, \alpha_2, \ldots, \alpha_n$ to be an enumeration of $Form$. Let

$$\begin{aligned} H_0 &= H \\ H_i &= \begin{cases} \text{if } H_{i-1} \cup \{\alpha_i\} & H_{i-1} \cup \{\alpha_i\} \in FSAT \\ \text{if } H_{i-1} \cup \{\neg\alpha_i\} & H_{i-1} \cup \{\neg\alpha_i\} \in FSAT \end{cases} \end{aligned}$$

Claim: Either $H_{i-1} \cup \{\alpha_i\}$ or $H_{i-1} \cup \{\neg\alpha_i\}$ is $FSAT$.

Proof: Suppose the contrary. Then for some finite subset X of H_{i-1} we have $X \cup \alpha_i$ is not SAT. Also, for some finite subset Y of H_{i-1} we have $Y \cup \neg\alpha_i$ is not SAT. Now, since $X \cup Y \subseteq H_{i-1}$, which is $FSAT$, $X \cup Y$ is satisfiable. So consider some $\tau \models X \cup Y$. Either $\tau \models \alpha_i$ or $\tau \models \neg\alpha_i$. So either $\tau \models X \cup \alpha_i$ or $\tau \models Y \cup \neg\alpha_i$. (contradiction). So the claim holds. □

We know that

$$H_0 \subseteq H_1 \subseteq H_2 \dots$$

Let

$$H' = \bigcup_{i \in \mathbb{N}} H_i$$

H' is exhaustive: Consider any $\varphi \in Form$ such that $\varphi = \alpha_i$. Either $\alpha_i \in H_i$ or $\neg\alpha_i \in H_i$.

H' is $FSAT$: Suppose it isn't. Then there exists a finite subset X of H' such that X is not satisfiable. Let $X = \beta_1, \dots, \beta_n$ where each β_i was either in H_0 or it was added at some later point j_i. Take $k = max(0, j_1, \dots, j_n)$. Then $X \subseteq H_{j_n}$. But we know that each H_i is $FSAT$, so this is impossible. □

Therefore, H' is SAT. Since H' is a superset of H, H is also SAT. □

From the Compactness Theorem, we may then conclude that H is unsatisfiable iff H is refutable.