COMP 409, Lecture 2

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1 Formulas

We wish to distinguish between expressions and formulas

Definition 1 Expression *An expression is any sentence constructed with the words (,),* $\leftrightarrow, \rightarrow, \wedge, \vee, \neg, p_0, p_1 \ \ldots$

Definition 2 Formulas *The set* **Form** *of formulas is the smallest set of expressions such that:*

1. *Prop* $\subseteq$ *Form*
2. *Closure:*
   (a) *If* $\varphi \in$ *Form, then* $(\neg\varphi) \in$ *Form*
   (b) *If* φ, $\psi \in$ *Form, then* $(\varphi \circ \psi) \in$ *Form*

In order for this to be a proper definition, we must argue that the smallest set is well-defined. The smallest set is the intersection of all sets of expressions that contain Prop *and satisfy the closure property.*

Alternatively, we may use the define **Form** inductively:

- $Form_0 =$ Prop (the set of all propositions)
- $Form_{n+1} = Form_n \cup \{(\neg\varphi)\colon\ \varphi \in Form\} \cup \{(\varphi \circ \psi)\colon\ \varphi,\ \psi \in Form\}$.

Lemma 1 $Form = \bigcup_{i=0}^{\infty} Form$

Proof Let $Form' = \cup_{i=1}^{\infty} Form_i$. We must show the following:

1. $Form \subseteq Form'$.
   It suffices to show that 1 and 2 in the definition of $Form$ are also satisfied by $Form'$. Clearly, $Prop \subseteq Form'$, since $Prop \subseteq Form_0$. As for the closure property, suppose that $\alpha, \beta \in Form'$, then $\alpha \in Form_i$ and $\beta \in Form_j$ for some $i, j \geq 0$. Let $k = \max\{i, j\}$. Clearly, $\alpha, \beta \in Form_k$. Thus, $(\neg\alpha) \in Form_{k+1} \subseteq Form'$ and $(\alpha \circ \beta) \in Form_{k+1} \subseteq Form'$. Thus, by definition, we have that $Form \subseteq Form'$.

2. $Form' \subseteq Form$.
   We prove this by showing, by induction on i, that $Form_i \subseteq Form$.

   Base: $Form_0 = Prop \subseteq Form$

   Induction Suppose $Form_i \subseteq Form$. By definition, $Form_{i+1} = Form_i \cup \{(\neg\alpha) : \alpha \in Form_i\} \cup \{(\alpha \circ \beta) : \alpha, \beta \in Form_i\}$. If $\alpha \in Form_i$, then $\alpha \in Form$, so $(\neg\alpha) \in Form$ by the closure property. Hence $\{(\neg\alpha) : \alpha \in Form_i\} \subseteq Form$. If $\alpha, \beta \in Form_i$, then $\alpha, \beta \in Form$, so $(\alpha \circ \beta) \in Form$. Hence $\{(\alpha \circ \beta) : \alpha, \beta \in Form_i\} \subseteq Form$ and $Form_{i+1} \subseteq Form$.$\blacksquare$

2 Parsing

How would you parse the statement ((p→ (q→r)) → ((p→q) → (p→r)))?

Definition 3 *Formulas are either atomic or composite*

- *For a composite formula $(\neg\varphi)$*
  - *$\neg$ is a primary connective*
  - *φ is a immediate subformula*
- *For a composite formula $(\varphi \circ \psi)$*
  - *$\circ$ is a primary connective*
  - *φ, ψ are immediate subformulas*

We would like to show that our language is not ambiguous

Theorem 1 Unique Readability *Every composite formula has a unique primary connective and immediate subformulas*

To prove this, we first need the following definition:

Definition 4 Prefix *An expression α is a prefix of β if $\beta = \alpha\gamma$. α is a strict prefix if $\gamma \neq \epsilon$. α is a nonempty prefix if $\alpha \neq \epsilon$.*

Proof of Unique Readability

- Let φ be $(\neg\psi)$. Then φ cannot be atomic.
  - Suppose φ is also $(\neg\psi\prime)$. Then $\psi = \psi\prime$ so ψ is unique.
  - Suppose φ is also $(\theta \circ \psi\prime)$. Then θ starts with $\neg$, which is impossible, since θ starts with (
- Let φ be $(\theta \circ \psi)$. As before, φ cannot be atomic, and cannot be $(\neg\psi\prime)$. Suppose φ is $(\theta\prime \circ \prime\psi\prime)$. Then either $\theta = \theta\prime, \circ = \circ\prime$,and $\psi = \psi\prime$, or θ is a prefix$\neq \theta\prime$. We would like to show this impossible.

If we can show that a strict prefix of a formula cannot itself be a formula, then we can finish our proof of Unique Readability. Before we proceed, we need a few more definitions:

Definition 5 l and r

- *l(φ) = number of (in φ*
- *r(φ) = number of) in φ*

Lemma 2 *φ a formula $\rightarrow$ l(φ) = r(φ)*

Proof of lemma

- **Base** φ is atomic, l(φ) = r(φ) = 0
- **Inductive**
  - φ is $(\neg\psi)$. Then l(φ) = 1 + l(ψ) = 1 + r(ψ) = r(φ)
  - φ is $(\theta \circ \psi)$. Then l(φ) = 1 + l(θ) + l(ψ) = 1 + r(θ) + r(ψ) = r(φ)

Lemma 3 *If α is a nonempty strict prefix of a formula φ, then l(α)$\geq$r(α)*

Corollary 1 *A strict prefix of a formula is not a formula*

Corollary 2 *Unique Readability*

Proof of lemma
Base: Trivial
Induction:

1. φ is $(\neg\psi)$
   (a) α is (: trivial
   (b) α is $(\neg$: trivial
   (c) α is $(\neg\psi\prime$ where $\psi\prime$ is a strict prefix of ψ Then l(α) = 1 + l($\psi\prime$) > 1 + r($\psi\prime$) > r($\psi\prime$) = r(α)
   (d) *alpha* is $(\neg\psi)$ Then l(α) = 1 + l(ψ) > 1 + r(ψ) > r(ψ) = r(α)
2. φ is $(\theta \circ \psi)$
   (a) $\theta = ($: trivial
   (b) $\theta = (\psi_1'$, where ψ_1' is a strict non-empty prefix of ψ_1: $\ell(\theta) = 1 + \ell(\psi_1') > 1 + r(\psi_1') > r(\theta)$, as desired.
   (c) $\theta = (\psi_1$: $\ell(\theta) = 1 + \ell(\psi_1) = 1 + r(\psi_1) > r(\theta)$, as desired.
   (d) $\theta = (\psi_1\circ$: same as $\theta = (\psi_1$.
   (e) $\theta = (\psi_1 \circ \psi_2'$, where ψ_2' is a strict non-empty prefix of ψ_2: $\ell(\theta) = 1 + \ell(\psi_1) + \ell(\psi_2') > 1 + r(\psi_1) + r(\psi_2') > r(\theta)$, as desired.

(f) $\theta = (\psi_1 \circ \psi_2$: $\ell(\theta) = 1 + \ell(\psi_1) + \ell(\psi_2) = 1 + r(\psi_1) + r(\psi_2) > r(\theta)$, as desired.

We now know enough to find the primary connective of any formula φ

Theorem 2 *The primary connective of a formula φ is the leftmost connective $\circ$ such that $\varphi = \alpha \circ \beta$ and l(α) = r(α) + 1*

Proof of theorem

1. φ is $(\neg\theta)$. Indeed, first negation is primary connective
2. φ is $(theta \circ \psi)$. Suppose $\theta = \alpha \circ\prime \beta$. Then $\alpha \neq \epsilon$ so l("(α") > r("(α") and $\circ\prime$ is not primary.