Comp 409: Lecture 20

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1 Expressibility

Given a Graph $G = (V, E)$ we define paths and reachability as follows:

Definition 1 *A path from $u \in V$ to $v \in V$ is a sequence $u_0, ..., u_k k \geq 0$ of nodes such that $u_0 = u$, $u_k = v$, and $(u_{i+1}) \in E$ for $0 \leq i \leq k-1$.*

Definition 2 *A vertex x is reachable from a vertex y if there exists a path from x to y. We write $reach(x, y)$.*

We would like to express $reach(x, y)$ in first-order logic. We can express concepts such as triangle free in first-order logic:

$$(\forall u, v, w)\neg(edge(u, v) \wedge edge(v, w) \wedge edge(w, u))$$

We might attempt to express $reach(x, y)$ in first order logic doing something like this:

$$reach_1(x, y) = E(x, y)$$
$$reach_i(x, y) = (\exists z)(E(x, z) \wedge reach_{i-1}(z, y))$$

This approach will not work since it requires an infinite number of statements. In fact, we will show that it is impossible to express *reach* in first-order logic.
It makes sense that we might not be able to express some properties in a parciular language, because choosing a language restricts what we are able to express. For instance, we cannot distinguish between two graphs that differ only in the shape of their edges if our language does not capture edge shape.

2 Isomorphism and Expressiveness

First we will define homomorphism.

Definition 3 *Let* $A = (D^A, P^A, ..., f^A)$ *and* $B = (D^B, P^B, ..., f^B)$.
A mapping $h : D^A \to D^B$ *is a homomorphism if*

1. $h(f^A(a_1, ..., a_k)) = f^B(h(a_1), ..., h(a_k))$
2. $P^A(a_1, .., a_k) \leftrightarrow P^B(h(a_1), ..., h(a_k))$

Definition 4 *h is additionally an* isomorphism *if it is 1-1 and onto.*

If h is an isomorphism then we say that A is isomorphic to B, denoted $A \simeq B$.

Theorem 1 *Let A, B be strcutures on the same vocabulary. Let ϕ be a sentence. If $A \simeq B$ then $A \models \phi$ iff $B \models \phi$.*

Proof: Let $A = (D^A, P^A, f^A)$ and $B = (D^B, P^B, f^B)$. Assume $A \simeq B$. Given that $A \models \varphi$, we must show that $B \models \varphi$. The induction hypothesis is that for an arbitrary formula θ we have that $A, \alpha \models \theta$ iff $B, h \circ \alpha \models \theta$.

- *Base Case:* θ is $P(t_1 \cdots t_n)$. $A, \alpha \models P(t_1 \cdots t_n)$
  iff $P^A(\overline{\alpha}(t_1) \cdots \overline{\alpha}(t_n))$
  iff $P^B(h(\overline{\alpha}(t_1)) \cdots h(\overline{\alpha}(t_n)))$
  iff (by an upcoming lemma) $P^B(\overline{h \circ \alpha}(t_1) \cdots \overline{h \circ \alpha}(t_n))$
  iff $B, h \circ \alpha \models P((t_1 \cdots t_n)$.
- *Inductive Case:*
  - $\neg, \wedge$: $A, \alpha \models \neg\psi$
    iff $B, h \circ \alpha \models \neg\psi$
    iff $A, \alpha \not\models \psi$
    iff $B, h \circ \alpha \not\models \psi$.
    Similarly for $\wedge$.
  - $\theta = \forall x \psi$
    $A, \alpha \models \forall x \psi$
    iff for all $a \in D^A$, $A, \alpha[x \mapsto a] \models \psi$
    iff for all $a \in D^A$, $B, h \circ (\alpha[x \mapsto a]) \models \psi$

iff for all $a \in D^A$, $B, (h \circ \alpha)[x \mapsto h(a)]) \models \psi$
iff for all $b \in D^B$, $B, (h \circ \alpha)[x \mapsto b] \models \psi$
iff $B, h \circ \alpha \models \forall x \psi$.

– $\varphi = \exists x \psi$ is similar.

Lemma 1 $h \circ \overline{\alpha} = \overline{h \circ \alpha}$ *(i.e., need to show for a term t, $h(\overline{\alpha}(t)) = \overline{h \circ \alpha}(t)$)*

Proof:

- *Base Case:* t is x: $h(\overline{\alpha}(x)) = h(\alpha(x)) = h \circ \alpha(x) = \overline{h \circ \alpha}(x)$
- *Inductive Case:* t is $f(t_1 \cdots t_n)$:$h(\overline{\alpha}(f(t_1 \cdots t_n))) = h(f^A(\overline{\alpha}(t_1) \cdots \overline{\alpha}(t_n))) = f^B(h(\overline{\alpha}(t_1)), \cdots, h(\overline{\alpha}(t_n))) = f^B(\overline{h \circ \alpha}(t_1) \cdots \overline{h \circ \alpha}(t_n)) = \overline{h \circ \alpha}(f(t_1 \cdots t_n))$.

3 Satisfiability and Validity

Definition 5 *A first-order formula ϕ is* satisfiable *if there is a structure A and a virable assignment α such that $A, \alpha \models \phi$.*
A first-order formula ϕ is valid *if for every structure A and variable assignment α, we have $A, \alpha \models \phi$.*