Lecture ??: COMP 409

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1 Satisfiability

Recall the satisfiability problem – given φ in first order logic, is φ satisfiable? This problem is equivalent to determing the validity of $\neg\varphi$. From previous lectures, we can transform φ in several different ways to force it to have a relatively simple structure:

1. Eliminate $\approx$ – preserves satisfiability.
2. Convert to prenex-normal form – preserves semantics.

Next we will show how to eliminate $\exists$ while preserving satisfiability. In this way, the sentences have a fairly simple syntax, i.e., $\forall$ is the only quantifier.

2 Skolemization

Skolemization is the process of eleminating existential qunatifiers from a formula in prenex normal form. We say that a formula $\varphi_\forall$ is in *skolem normal form* if it is of the form $\forall y_1 \ldots \forall y_n \theta$, where θ is quantifier-free formula. This transformation preserves satisfiability: for every sentence φ, there exists a universal sentence $\varphi_\forall$ such that φ is satisfiable iff $\varphi_\forall$ is satisfiable. We begin with a simple case.

Theorem 1 *$\forall x\, \forall y\, \theta$ is satisfiable iff $\forall x\, \theta[x \mapsto f(x)]$ is satisfiable, where $f(x)$ is defined in the proof.*

Proof. Let A such that $A \models \forall x\, \forall y\, \theta$. Define $f : D^A \mapsto D$ such that $f^A(a) = b$ where $A, [x \mapsto a, y \mapsto b] \models \theta$. Since b may not be unique, we invoke the Axiom of Choice to choose a particular b.

Obtain the structure A' by adding to A the function f defined above. We claim that $A' \models \forall x\, \theta[y \mapsto f(x)]$. To proof the claim, we need to show that for all $a \in D^A$, $A', [x \mapsto a] \models \theta[y \mapsto f(x)]$. We know that $A', [x \mapsto a, y \mapsto b] \models \theta$, where $b = f^A(a)$. By a simple inductive argument, it can be shown that $A', [x \mapsto a] \models \theta[y \mapsto f(x)]$ iff $A, [x \mapsto a, y \mapsto b] \models \theta$. For the other way around, that is to find $\exists y$, choose y to be $f(x)$. $\square$

This transformation can be easily generalized. Suppose $\phi = \forall x_1 \ldots \forall x_n \exists y \theta$ and let A be a structure such that $A \models \phi$. Then we know that for every m-tuple $\langle a_1, \ldots, a_n \rangle \in (D^A)^n$ there exists a $b \in D^A$ such that $A \models \theta[\alpha]$ where the assignment α maps x_i to a_i and y to be. This allows us to construct a table that for every n-tuple it indicates the value of y that satisfies θ. In cases where y could have more than one value, we simply choose one of them.

Observe that this table maps each n-tuple in D^A to a unique element of D^A. Therefore, it defines a function $f^(n)$ on the domain of A. We now define a new structure A' which assoiciates the unique symbol f to the function defined above. Then our transformation replaces every occurrence of y with $f(x_1, \ldots, x_n)$.

3 Herbrand Universes and Structures

If ϕ is satisfiable, how can we find a structure A that satisfies ϕ? We will use the syntax as our domain.

Consider a vocabulary which contains at least one constant symbol.

Definition 1 *A* ground term *is a term containing no variables.*

Definition 2 *The* Herbrand Universe *(denoted H) of a language, is the set of ground terms in a language.*

Definition 3 *A* Herbrand Structure *of a formula φ is a structure $\mathcal{H}$ such that $D^{\mathcal{H}}$ is the Herbrand Universe of φ and $f^{\mathcal{H}}([t_1] \ldots [t_n]) = [f(t_1 \ldots t_n)]$.*

I.e.,

$$\mathcal{H} = (D^{\mathcal{H}}, P_1^{\mathcal{H}} \ldots P_k^{\mathcal{H}}, f_1^{\mathcal{H}} \ldots f_\ell^{\mathcal{H}})$$

where $D^{\mathcal{H}}$ is an *Herbrand Universe.* Terms are now both syntax and semantics: $f_i^{\mathcal{H}}(\text{“}t_1\text{”}, \ldots, \text{“}t_n\text{”}) = \text{“}f_i(t_1, \ldots, t_n)\text{”}$.

Definition 4 *An* Herbrand Model $\mathcal{H}$ *of a sentence φ is an Herbrand structure that satisfies φ.*

Theorem 2 (Herbrand's Theorem) *If ϕ is a satisfiable sentence, then it has an Herbrand model.*

Conclusion Every Herbrand model is countably infinite.

Corollary 1 (Löwenheim-Skolem Theorem) *If φ is satisfiable, then it has a countable model.*

Corollary 2 *There is no FOL sentence that say "There are uncountably many elements."*

4 Validity is Recursively Enumerable

Definition 5 *Let ϕ be a universal sentence $\forall x_1 \ldots \forall x_n \theta$. Then,*

$$ground(\phi) = \{\theta[x_1 \mapsto t_1, \ldots, x_n \mapsto t_n] : t_i \in H\}.$$

Note that elements of $ground(\phi)$ have no variables.

Lemma 1 *Let A be an Herbrand structure. Then $A \models \phi$ iff $A \models ground(\phi)$.*

Proof. $A \models \phi$ iff for all $t_1, \ldots, t_k \in H$ we have $A, [x_1 \mapsto t_1, \ldots, x_k \mapsto t_k] \models \phi$. By a simple inductive argument on the structure of ϕ we can show that $A, [x_1 \mapsto t_1, \ldots, x_k \mapsto t_k] \models \phi$ iff $A \models \phi[x_1 \mapsto t_1, \ldots, x_n \mapsto t_n]$. □

Definition 6 *For every k-ary relation symbol P and $t_1, \ldots, t_k \in H$, we introduce a proposition $P_{t_1,\ldots,t_k}$.*

This gives us the following result:

Lemma 2 *$ground(\phi)$ is satisfiable in an Herbrand structure iff $ground(\phi)[p(t_1, \ldots, t_k) \mapsto p_{t_1,\ldots,t_k}$ is satisfiable.*

For example, $p(f(c)) \wedge \neg p(g(d))$ becomes $P_{f(c)} \wedge \neg P_{g(d)}$.

Thus, φ is unsatisfiable iff $ground(\varphi)$ is unsatisfiable. It follows from the Compactness Theorem for first-order logic, that some finite subset of $ground(\varphi)$ is unsatisfiable. In this way, we have reduced a formula in first order logic to one in propositional logic. In conclusion, to determine the unsatisfiability of φ, we enumerate all finite subsets of $ground(\phi)$ and check for unsatisfiability. Finally, recall that φ is valid iff $\neg\varphi$ is unsatisfiable.

Theorem 3 (Gödel's Completeness Theorem) *Validity for first order logic is recursively enumerable.*