1 Lifting

A relation $R \subseteq A \times B$ can be lifted by defining another relation $R' \subseteq 2^A \times 2^B$.

A Universal lift is a lift if for $(U, V) \in R'$ we have that $\forall a \in U$, $\forall b \in V$ we have that the pair $(a,\ b)$ is in R.

In the past we've seen: $\tau \in 2^{Prop}$ and $\varphi \in Form$. These apply to single items, and can be overloaded to instead apply to sets: $T \subseteq 2^{Prop}$ and $\Phi \subseteq Form$. Therefore, the relation $\tau \models \varphi$ (single items) can be overloaded to $T \models \Phi$ (sets).

Definition 1 *Let $T \subseteq 2^{Prop}$ and $\Phi \subseteq Form$. $T \models \Phi$ if $\forall \tau \in T$, $\forall \varphi \in \Phi$, $\tau \models \varphi$*

Remember that $models(\varphi)$ is the set of truth assignments in which φ is true.

Definition 2 *Let $\Phi \subseteq Form, models(\Phi) = \cap_{\varphi \in \Phi} models(\varphi)$*

Lemma 1 *$T \models \Phi$ iff $T \subseteq models(\Phi)$*

By lifting we change from membership ($\in$) to subset ($\subseteq$).

Definition 3 *Let $\Phi, \Psi \subseteq Form, \Phi \models \Psi$ if $models(\Phi) \subseteq models(\Psi)$*

In the future we will see the relationship between $\Phi \models \Psi$ and $\varphi \to \psi$.

2 Classifying formulas

Definition 4 *We can classify any formula ψ as:*

1. *ψ is valid or a tautology if $\forall \tau \in 2^{Prop}$, $\tau \models \psi$*
2. *ψ is a contradiction if $\forall \tau \in 2^{Prop}$, $\tau \not\models \psi$*
3. *ψ is satisfiable if $\exists \tau \in 2^{Prop}$, $\tau \models \psi$*

2.1 Philosopher

Interested in valid, or formulas that are always true.

2.2 Electrical Engineer

Interested in satisfiable and not valid. They are not interested in circuits stuck in 1 (valid) or 0 (contradiction).

2.3 Software Engineer

Again, they are interested in satisfiable and not valid. They are not interested in formulas that are always true or false.

2.4 Relations between classifications

If ψ is valid, $(\neg\psi)$ is contradiction.
If ψ is not a contradiction, ψ is satisfiable.
If ψ is not satisfiable, ψ is a contradiction.

Lemma 2 *Classification of negated formulas*

1. *ψ is valid iif $(\neg\psi)$ is not satisfiable.*
2. *ψ is not valid iff $(\neg\psi)$ is satisfiable.*
3. *ψ is satisfiable iff $(\neg\psi)$ is not valid.*

Proof:

1. ψ is valid iff $\forall\tau \in 2^{Prop}\ \tau \models \psi$ (defn)
   $\forall\tau \in 2^{Prop}\ \tau \models \psi$ iff $\tau \not\models (\neg\psi)$ (defn)
   $\forall\tau \in 2^{Prop}\ \tau \models \psi$ iff $\forall\tau \in 2^{Prop}\ \tau \not\models (\neg\psi)$ (combine defns)
   iff $(\neg\psi)$ is not satisfiable.
2. ψ is not valid iff $\exists\tau \in 2^{Prop}\ \tau \not\models \psi$
   $\exists\tau \in 2^{Prop}\ \tau \not\models \psi$ iff $\tau \models (\neg\psi)$
   $\exists\tau \in 2^{Prop}\ \tau \not\models \psi$ iff $\exists\tau \in 2^{Prop}\ \tau \models (\neg\psi)$ iff $(\neg\psi)$ is satisfiable
3. Similar, not going to prove in class

∎

3 Relationship between $\models$ and $\rightarrow$

Lemma 3 *$\psi \models \varphi$ iff $(\psi \rightarrow \varphi)$ is valid*
Proof:

- *Suppose $\psi \models \varphi$, let $\tau \in 2^{Prop}$*
  - *if $\tau \not\models \psi$, then $\tau \models (\psi \rightarrow \varphi)$*
  - *if $\tau \models \psi$, then $\tau \models \varphi$, thus $\tau \models (\psi \rightarrow \varphi)$*
- *Suppose $(\psi \rightarrow \varphi)$ is valid, let $\tau \in 2^{Prop}$*
  - *if $\tau \models \psi$, $\tau \models (\psi \rightarrow \varphi)$, thus $\tau \models \varphi$*

∎

4 Conventions

By convention: $models(\emptyset) = \bigcap_{\varphi \in \emptyset} models(\varphi) = 2^{Prop}$.

Let $\varphi \in Form$. What should $\emptyset \models \varphi$ mean?

- We can think of $\emptyset$ as the empty set of truth assignment, in which case the meaning is $\emptyset \subseteq models(\varphi)$, which surely holds.
- We can think of $\emptyset$ as the empty set of formulas, in which case the meaning is $models(\emptyset) \subseteq models(\varphi)$. By convention, $models(\emptyset) = 2^{Prop}$, so $\emptyset \models \varphi$ means that φ is *valid*.

$\models \varphi$ is shorthand to mean φ is valid.

5 Properties of $\models$

There are three fundamental properties that each relation is tested for : *reflexivity, symmetry, transitivity.* We will examine these three properties for $\models$.

1. Reflexivity: $\models$ is reflexive, because $models(\Phi) \subseteq models(\Phi)$.
2. Symmetry: $\models$ is not symmetric, because if $\Phi = p \wedge q$ and $\Psi = p$, then $\Phi \models \Psi$ but not $\Psi \models \Phi$. $\models$ is not antisymmetric either, since $\subseteq$ is not antisymmetric and $\models$ is defined as a $\subseteq$ relation between two sets.
3. Transitivity: $\models$ is transitive, because if $\Phi \models \Psi$ and $\Psi \models \Theta$, then $models(\Phi) \subseteq models(\Psi) \subseteq models(\Theta)$. Thus $\Phi \models \Theta$.

A relation that is reflexive, not symmetric and transitive is called *partial order* or *weak partial order*. From the above we can see that $\models$ is a weak partial order, because it is reflexive, not symmetric and transitive. Another weak partial order relation is the $\subseteq$.

6 Logical Equivalence

Definition 5 *$\Phi \models\!\!\!\!\!\dashv \Psi$ if $\Phi \models \Psi$ and $\Psi \models \Phi$.*

Lemma 4 *Logical equivalence is an equivalence relation true for both formulas and sets of formulas.*

Logical equivalence is written as $\models\!\!\!\!\!\dashv$, and is reflexive, symmetric, and transitive.

Just as $\psi \models \varphi$ is related to $(\psi \rightarrow \varphi)$, $\psi \models\!\!\!\!\!\dashv \varphi$ is related to $(\psi \leftrightarrow \varphi)$.

Lemma 5 *Let $\psi, \varphi \in Form$, $\psi \models\!\!\!\!\!\dashv \varphi$ iff $\models (\psi \leftrightarrow \varphi)$.*