Lecture 12

September 30, 2005

1 Review

In the last lecture, we looked at the (tentative) structure of the class **NP**. The class **P** is a subset of **NP**, but until now no one has been able to prove or deny whether $\mathbf{P} = \mathbf{NP}$. We also introduced the class **NP**-complete, which consists of the "hardest" **NP** problems. Any problem $P \in \mathbf{NP}$-complete has the property $Q \leq_p P$ for all $Q \in \mathbf{NP}$. Given a specific problem, one cannot tell if it belongs to $\mathbf{NP} - \mathbf{P}$. However, if we assume that $\mathbf{P} \neq \mathbf{NP}$, it is certain that all **NP**-complete problem are in $\mathbf{NP} - \mathbf{P}$.

The first **NP**-complete problem was SAT, and the proof that it is **NP**-complete is due to Cook and Levin. Cook and Levin proved that all **NP** problems are polynomially reducible to SAT. In this lecture, however, we only show how to reduce a specific problem, 3-COLOR, to SAT.

2 3-COLOR

Informally, the 3-COLOR problem asks if a graph can be colored in such a way that two nodes connected by an edge have different colors. Formally, given a graph $G = (V, E)$, does there exist a function $f : V \to \{1, 2, 3\}$, where 1, 2 and 3 represent "colors", such that $f(u) \neq f(v)$ if $(u, v) \in E$? If such a function exists, the graph G is said to be 3-*colorable*.

3-COLOR $\leq_p$ SAT

We first build a set of CNF clauses that represents the coloring constraints on the graph G. For each vertex $v \in V$, we define three propositions $p_{v,i}$, where $i \in \{1, 2, 3\}$. $p_{v,i}$ is 1 if vertex v's color is i and 0 otherwise. The condition that each vertex is colored with exactly one color $i \in \{1, 2, 3\}$ can be expressed as

$$\bigwedge_{v \in V} (p_{v,1} \vee p_{v,2} \vee p_{v,3}) \wedge \neg(p_{v,1} \wedge p_{v,2}) \wedge \neg(p_{v,2} \wedge p_{v,3}) \wedge \neg(p_{v,3} \wedge p_{v,1}). \quad (1)$$

And the requirement that two adjacent vertices have different colors is expressed as

$$\bigwedge_{(u,v) \in E} \neg(p_{u,1} \wedge p_{v,1}) \wedge \neg(p_{u,2} \wedge p_{v,2}) \wedge \neg(p_{u,3} \wedge p_{v,3}). \quad (2)$$

By using DeMorgan's laws, $\neg(p \wedge q) = (\neg p \vee \neg q)$. Therefore, the above clauses can be transformed to equivalent CNF clauses. Let φ_G denote the set of these CNF clauses. We now need to show that G is 3-colorable iff φ_G is satisfiable.

Let $c : V \to \{1, 2, 3\}$ be a 3-coloring of G. Define the truth assignment

$$\tau_c(p_{v,i}) = \begin{cases} 1 & \text{if } c(v) = i, \\ 0 & \text{otherwise.} \end{cases}$$

From the construction of φ_G, it is easy to see that $\tau_c \models G$.

Conversely, assume that τ is a truth assignment of φ_G. Define

$$\begin{array}{rcl} c_\tau : V & \to & \{1, 2, 3\} \\ c_\tau(v) & \mapsto & i \text{ for } \tau(p_{v,i}) = 1. \end{array}$$

Because the clauses in equation (1) is true under the assignment τ, every vertex $v \in V$ is colored with only one color. Because the clauses in (2) are also true, vertices v and u cannot have the same color if (v, u) is an edge of G. Therefore, c_τ is a 3-coloring of G.