Lecture 2

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1 Review from last lecture

Definition 1 **Form**: *The set* **Form** *of formulas is the intersection of all sets P of Expressions such that:*

1. *Prop* $\subseteq P$
2. *Closure: If* $\varphi, \psi \in P$, *then so are* $(\neg\varphi)$ *and* $(\varphi \circ \psi)$.

Where $\circ$ *represents generic binary connectives such as* $\wedge$ *and* $\vee$. *Prop is the set of all propositions.*

The previous definition of **Form** is non-constructive, which can make proofs about **Form** inconvenient. Alternatively, we can define **Form** inductively.

Definition 2 **Form'**:

1. $Form_0 = Prop$
2. $Form_{i+1} = Form_i \cup \{(\neg\varphi)\colon\ \varphi \in Form_i\} \cup \{(\varphi \circ \psi)\colon\ \varphi,\ \psi \in Form_i\}$.
3. *Form'* $= \bigcup_{i=0}^{\infty} Form_i$

Form will refer to the first definition and Form' will refer to the inductive definition. This inductive definition "builds from below." The question now is this: Are both of these definitions equivalent? The answer is yes, as is shown by the next lemma.

Lemma 1 *Form = Form'*

Proof We can prove that Form = Form', if we can show that Form $\subseteq$ Form' and Form' $\subseteq$ Form.

1. $Form \subseteq Form'$. We will show that $Form'$ satisfies the two key properties: contains PROP, and satisfies closure.
   - $Prop \subseteq Form_0, Form_i \subseteq Form_{i+1} \Rightarrow Prop \subseteq Form_i$
     Therefore $Prop \subseteq Form'$.

- Let $\varphi, \psi \in Form'$. From the way we build $Form'$ it follows that there must exist i and j, such that $\varphi \in Form_i$ and $\psi \in Form_j$. Let $k = max(i, j)$. Clearly $\varphi, \psi \in Form_k$.
  $(\neg\varphi) \in Form_{k+1} \Rightarrow (\neg\varphi) \in Form'$ and
  $(\varphi \circ \psi) \in Form_{k+1} \Rightarrow (\varphi \circ \psi) \in Form'$

  We have shown that $Form'$ satisfies closure and contains PROP. Since $Form$ is a subset of every set with these two properties, it follows that $Form \subseteq Form'$.

2. $Form' \subseteq Form$.
   We need to show that if P contains PROP and satisfies closure, then $Form' \subseteq P$. We do this by using induction on i.

   Base: $Form_0 = Prop \subseteq P$

   Induction:
   $Form_i \subseteq P$
   $Form_{i+1} = Form_i \cup \{(\neg\varphi) : \varphi \in Form_i\}$
   $\cup \{(\varphi \circ \psi) : \varphi, \psi \in Form_i\}$
   By closure: $Form_{i+1} \subseteq P$

∎

In mathematical writing, there is a tricky balance between rigor and readability. You can imagine a proof that is precise to every detail and can be checked by a machine. But if it looked too arcane to the human eye, it would be useless in easily conveying the information to the reader. (Think about trying to read assembly versus reading Java or OCAML). Of course, if a proof lacks the necessary details it can't be used for anything. This balance if something you should always be thinking about when you write proofs. The previous proof is a good example of how to prove simple things rigorously.

2 Parsing

Consider the formula $((p \to q) \to ((q \to r) \to (p \to r)))$. If you are given an expression, how can you tell if it is valid? Parsing lets us examine an expression to determine validity. The following will help us formalize this problem.

Definition 3 *Formulas are either atomic or composite. The set of atomic formulas = Prop.*

- *For a composite formula* $(\neg\varphi)$
  - $\neg$ *is the primary connective*
  - φ *is the immediate sub-formula*
- *For a composite formula* $(\varphi \circ \psi)$

- $\circ$ *is the primary connective*
- φ, ψ *are immediate sub-formulas*

We would like to show that our language is not ambiguous, i.e. that there is only one way to read a formula in FORM.

Theorem 1 Unique Readability *Every composite formula has a unique primary connective and well-defined immediate sub-formulas*

To prove this, we first need the following definition:

Definition 4 (Prefix) *An expression α is a prefix of β if $\beta = \alpha\gamma$. α is a* strict prefix *if $\gamma \neq \epsilon$. α is a nonempty prefix if $\alpha \neq \epsilon$.*

Proof of Unique Readability We prove this using structural induction.
Base: φ is an atomic proposition. This statement is trivially true.
Inductive Step: We must consider two cases.

1. Suppose φ is $(\neg\theta)$.
   - φ is not atomic
   - Suppose φ is also $(\neg\theta\prime)$. It follows that $\theta = \theta\prime$, since strings are a semi-group with left and right cancellation.
   - φ is also $(\alpha \circ \beta)$. So α must start with $\neg$. This is impossible, because a formula is either atomic or starts with (.
2. Suppose φ is $(\alpha \circ \beta)$.
   - φ is not atomic
   - φ is not $(\neg\theta)$.
   - Suppose φ is $(\theta \circ \prime\psi)$.
     - $\alpha = \theta \Rightarrow \circ = \circ\prime and \beta = \varphi$
     - α is a strict prefix of θ, or θ is a strict prefix of α. This is impossible by the next theorem.

If we can show that a strict prefix of a formula cannot itself be a formula, then we can finish our proof of Unique Readability. Before we proceed, we need a few more definitions:

Definition 5 Parenthesis Counting

- *l(φ) = number of '(' in φ*
- *r(φ) = number of ')' in φ*

Lemma 2 *φ a formula $\rightarrow$ l(φ) = r(φ)*

Proof of lemma

- **Base** $\varphi \in Prop \Rightarrow \mathrm{l}(\varphi) = \mathrm{r}(\varphi) = 0$
- **Inductive**
  - φ is $(\neg\theta)$. Then $\mathrm{l}(\varphi) = 1 + \mathrm{l}(\theta) = 1 + \mathrm{r}(\theta) = \mathrm{r}(\varphi)$
  - φ is $(\theta \circ \psi)$. Then $\mathrm{l}(\varphi) = 1 + \mathrm{l}(\theta) + \mathrm{l}(\psi) = 1 + \mathrm{r}(\theta) + \mathrm{r}(\psi) = \mathrm{r}(\varphi)$

In the next lecture, we'll show that if you stop reading a formula too early, you end up with too many '('s. The number of parenthesis is a distinguishing characteristic of formulas.