# COMP 409, Lecture 20

Friday, October 21, 2005

## 1 Review

The *vocabulary* of predicate calculus consists of predicate symbols (with arity), function symbols (with arity) and variables. The predicate symbols denote relations. The syntax of first order logic is defined on this vocabulary:

- Variables + Function symbols $\longrightarrow$ Terms
- Terms + Predicate symbols $\longrightarrow$ Formulas

Predicate calculus can be used to model mathematical structures. A relational structure or world $A$ consists of a tuple $(D, f_1^A, \ldots, f_n^A, P_1^A, \ldots, P_m^A)$ where

- $D$ is the universe or domain of 'atomic' objects.
- $f_i^A$ is the interpretation of $f_i$ in $A$.
- $f_i : D^k \rightarrow D$ where $k$ is the arity of $f_i$.
- $P_i^A$ is the interpretation of $P_i$ in $A$.
- $P_i \subseteq D^k$ where $k$ is the arity of $P_i$.

For example, we can model the mathematical concept of graphs as a pair $G = (V, E)$ where $V$ is the universe of vertices and $E$ is the binary edge relation.

## 2 Semantics of First-Order logic

Semantics relates the syntax to the world (relational structure). $A \models \varphi$ denotes that formula $\varphi$ is true in the world $A$. Here '$\models$' is the semantical relation.

Consider the statement $A \models (x = 2)$. Does it make sense to ask whether the formula '$x = 2$' is true in the world $A$? The truth of the formula depends on the value of $x$, but $x$ is a variable and can take any value. Some values of $x$ might make the formula true, while others might falsify it. Thus, we need to qualify our answer by saying that the formula is true when $x$ has a certain value. To do this we can use a function $\alpha$ that assigns values to variables. Such a function is called a 'binding'. Thus, statements about semantics have the form

$A, \alpha \models \varphi$ where $A$ is a structure, $\varphi$ is a formula and $\alpha$ is a variable assignment, i.e., $\alpha : Var \rightarrow D$. Note that our notion of truth is now a ternary relation, as opposed to the binary relation of propositional logic.

We next define this notion formally. First we extend the notion of bindings to terms.

**Definition 1** *Given structure $A$ and $\alpha : Var \rightarrow D$, we define $\overline{\alpha} : Term \rightarrow D$ as follows:*

1. $\overline{\alpha}(x) = \alpha(x)$*, where $x$ is a variable*
2. $\overline{\alpha}(f(t_1, \ldots, t_k)) = f^A(\overline{\alpha}(t_1), \ldots, \overline{\alpha}(t_k))$*, where $k$ is the arity of $f$, and* $t_1, \ldots, t_k \in \textsc{Terms}$.

Note that if $f$ is a 0-ary function then $\overline{\alpha}(f) = f^A$. We now define $A, \alpha \models \varphi$. (note: other notations in use include $A \models_\alpha \varphi$ and $A \models \varphi[\alpha]$).

1. $A, \alpha \models P(t_1, \ldots, t_k)$ if $(\overline{\alpha}(t_1), \ldots, \overline{\alpha}(t_k)) \in P^A$.
2. $A, \alpha \models (\theta \wedge \psi)$ if $A, \alpha \models \theta$ and $A, \alpha \models \psi$.
   $A, \alpha \models (\neg\theta)$ if $A, \alpha \not\models \theta$.
3. $A, \alpha \models (\exists x)\varphi$ if there is some $a \in D$ such that $A, \alpha[x \mapsto a] \models \varphi$.
   $A, \alpha \models (\forall x)\varphi$ if for all $a \in D$ we have $A, \alpha[x \mapsto a] \models \varphi$.

A note about quantifiers and their ordering: The order of quantifiers does not matter if they are of the same type. For example, $(\forall x)(\forall y)P(x, y)$ is the same as $(\forall y)(\forall x)P(x, y)$. The order does matter when quantifiers of different types are mixed. For example, $(\forall x)(\exists y)P(x, y)$ is not the same as $(\exists y)(\forall x)P(x, y)$. As an aside, mixed quantifiers can lead to ambiguities in natural language. For example, consider the following sentence:

'There is a mother to all children.'

The above sentence can be read to have two different meanings:

- There exists $x$, such that for every child $y$, $x$ is a mother to $y$.
- For every child $y$, there exists $x$, such that $x$ is a mother to $y$.

In the first instance, the sentence can be read to mean that every child has the same mother, while the second one implies the weaker condition that every child has a mother.