Lecture 23

October 28, 2005

1 Expressibility

Recall that last time we modeled sequential circuits with first order logic. We can also think of these circuits as a graph where vertices are states and edges are transitions between the states. In our first-order logic model of the sequential circuit, we ran into difficulty expressing the notion of whether certain states were reachable. Let's think about this problem in terms of a graph.

Given a Graph $G = (V, E)$ we define paths and reachability as follows:

Definition 1 (Paths). *A path from $u \in V$ to $v \in V$ is a sequence $u_0, \ldots, u_k$ of $k \geq 0$ nodes such that $u_0 = u$, $u_k = v$, and $(u_i, u_{i+1}) \in E$ for $0 \leq i \leq k-1$.*

Definition 2 (Reachability). *A vertex x is* reachable *from a vertex y if there exists a path from x to y. We write $reach(x, y)$.*

We would like to express $reach(x, y)$ in first-order logic. Consider:

$$\begin{aligned} reach_1(x, y) &= E(x, y) \\ reach_i(x, y) &= (\exists z)(E(x, z) \wedge reach_{i-1}(z, y)) \end{aligned}$$

This approach will not work for arbitrary graphs since it may require an infinite number of statements. In fact, we will show that it is impossible to express *reach* in first-order logic.

It makes sense that we might not be able to express some properties in a particular language, because choosing a language restricts what we are able to express. For instance, we cannot distinguish between two graphs that differ only in the shape of their edges if our language does not capture edge shape.

2 Isomorphism and Expressiveness

Consider a formula φ expressed in the standard vocabulary and syntax of first-order-logic. Now consider two worlds that model φ. At times, these two worlds are so similar that we can think of them as indistinguishable and therefore interchangeable.

We now capture this intuition formally. How do we show that two worlds are similar? One way is by constructing a map between them.

Definition 3 (Homomorphism). *Let* $A = (D^A, P^A, \ldots, f^A)$ *and* $B = (D^B, P^B, \ldots, f^B)$. *A mapping* $h : D^A \to D^B$ *is a* homomorphism *if*

1. $h(f^A(a_1, \ldots, a_k)) = f^B(h(a_1), \ldots, h(a_k))$
2. $P^A(a_1, \ldots, a_k) \leftrightarrow P^B(h(a_1), \ldots, h(a_k))$

It would also be nice if a homomorphic map went both ways.

Definition 4 (Isomorphism). *A homomorphism* h *is additionally an* isomorphism *if it is one-to-one and onto.*

If h is an isomorphism then we say that A is isomorphic to B, denoted by $A \simeq B$.

Theorem 1. *Let* A, B *be structures on the same vocabulary and let* α *be an assignment on* A. *Let* φ *be a formula. If* $A \simeq B$ *then* $A, \alpha \models \varphi$ *iff* $B, (h \circ \alpha) \models \varphi$.

Before the main proof, we first need some help.

Lemma 1. *Let* h *be a homomorphism. Then* $h \circ \overline{\alpha} = \overline{h \circ \alpha}$. *In other words, for a term* t, $h(\overline{\alpha}(t)) = \overline{h \circ \alpha}(t)$.

Proof.

- *Base Case:* t is x:

$$h(\overline{\alpha}(x)) = h(\alpha(x)) = h \circ \alpha(x) = \overline{h \circ \alpha}(x)$$

- *Inductive Case:* t is $f(t_1 \ldots t_n)$:

$$\begin{aligned} h(\overline{\alpha}(f(t_1 \ldots t_n))) &= h(f^A(\overline{\alpha}(t_1) \ldots \overline{\alpha}(t_n))) \\ &= f^B(h(\overline{\alpha}(t_1)), \ldots, h(\overline{\alpha}(t_n))) \\ &= f^B(\overline{h \circ \alpha}(t_1) \ldots \overline{h \circ \alpha}(t_n)) \\ &= \overline{h \circ \alpha}(f(t_1 \ldots t_n)) \end{aligned}$$

$\square$

Now to the main proof.

Proof: Let $A = (D^A, P^A, \ldots, f^A, \ldots)$ and $B = (D^B, P^B, \ldots, f^B, \ldots)$. Assume that $A \simeq B$. Given that $A, \alpha \models \varphi$, we must show that $B, (h \circ \alpha) \models \varphi$.

We will use induction for the proof. The induction hypothesis is that for an arbitrary formula θ we have that $A, \alpha \models \theta$ iff $B, h \circ \alpha \models \theta$.

- *Base Case:* θ is $P(t_1 \ldots t_n)$.

  $A, \alpha \models P(t_1 \ldots t_n)$
  iff $P^A(\overline{\alpha}(t_1) \ldots \overline{\alpha}(t_n))$
  iff $P^B(h(\overline{\alpha}(t_1)) \ldots h(\overline{\alpha}(t_n)))$
  iff $P^B(\overline{h \circ \alpha}(t_1) \ldots \overline{h \circ \alpha}(t_n))$ [by the lemma]
  iff $B, (h \circ \alpha) \models P((t_1 \ldots t_n)$.

- *Inductive Case:*
  - $\neg, \wedge$: $A, \alpha \models \neg\psi$
    iff $B, h \circ \alpha \models \neg\psi$
    iff $A, \alpha \not\models \psi$
    iff $B, h \circ \alpha \not\models \psi$.
    Similarly for $\wedge$.
  - $\theta = \forall x \psi$
    $A, \alpha \models (\forall x)\psi$
    iff for all $a \in D^A$, $A, \alpha[x \mapsto a] \models \psi$
    iff for all $a \in D^A$, $B, h \circ (\alpha[x \mapsto a]) \models \psi$
    iff for all $a \in D^A$, $B, (h \circ \alpha)[x \mapsto h(a)]) \models \psi$
    iff for all $b \in D^B$, $B, (h \circ \alpha)[x \mapsto b] \models \psi$
    iff $B, (h \circ \alpha) \models (\forall x)\psi$.
  - $\varphi = \exists x \psi$ is similar.

This concludes the proof.