Lecture 24

October 31, 2005

1 Introduction and Motivation

Recall that in the last lecture we proved the following theorem:

Theorem 1 (Isomorphism Theorem) *Let A, B be structures on the same vocabulary. Let φ be a formula. If $A \simeq B$ then $A \models \varphi$ iff $B \models \varphi$.*

In this lecture we will use this theorem to partition the set of all structures. But first, some definitions. Notice the similarities between these definitions and the corresponding ones in propositional logic.

Definition 1 (Satisfiability) *A first-order formula φ is* satisfiable *if there is a structure A and a variable assignment α such that $A, \alpha \models \varphi$.*

Definition 2 (Validity) *A first-order formula φ is* valid *if for every structure A and variable assignment α, we have $A, \alpha \models \varphi$.*

We start by defining a notion of how "similar" two formulas are.

2 Elementarily Equivalence

Definition 3 (Elementary Equivalence) *Let A, B be structures on some vocabulary. We say that A and B are* elementarily equivalent*, denoted by $A \equiv B$, if for all sentences φ we have $A \models \varphi$ iff $B \models \varphi$.*

Notice that $A \simeq B \Rightarrow A \equiv B$. In other words, isomorphism is a stronger notion of equivalence than elementary equivalence. Symbolically, this can be expressed as $\simeq \subseteq \equiv$.

Theorem 2 *Let Φ be a set of sentences. Let models(Φ) $= \{A : A \models \Phi\}$. Then models(Φ) is a union of equivalence classes of $\equiv$.*

3 Elementary and Extended Elementary Classes

There is one very pragmatic question that we would like to answer: what classes of structures can we describe using first-order sentences? For example, can we describe the class of all connected graphs? Before we try to answer these questions, we make an important distinction between classes that can be expressed using a single formula, and classes that can be expressed using infinitely many formulas.

Definition 4 (*EC*) *A class C of structures is* elementary*, denoted* EC*, if $C = models(\varphi)$ for some sentence φ.*

Examples of elementary structures include the set of triangle-free graphs, graphs with transitivity, and others

Definition 5 (EC_Δ) *A class C of structures is* extended elementary*, denoted EC_Δ, if $C = models(\Phi)$ for some set Φ of sentences.*

In order to give an example for a class of structures that are in EC_Δ, recall the following definition of k-colorable graphs.

Definition 6 (k-colorability) *$G = \{V, E\}$ is k-colorable iff there is a mapping $c : V \mapsto \{1, \ldots, k\}$ such that $\langle u, v\rangle \in E \rightarrow c(u) \neq c(v)$.*

In the remainder of this lecture we will use $2C$ to denote the class of 2-colorable graphs.

Question 1 *Is 2C EC? Or is it EC_Δ?*

Lemma 1 *A graph is 2C iff it has no cycle of odd length.*

This lemma can be proved easily by induction. The interested student is encouraged to write down the proof and discuss it with the TA during office hours.

Theorem 3 *2C is EC_Δ.*

Proof: (This is a well-known result from graph theory.)

We can define a set of first-order sentences which model 2-colorable graphs. Define $p_n(u, v)$ recursively as follows:

$$p_1(x, y) = \langle x, y\rangle \in E \tag{1}$$
$$p_n(x, y) = (\exists z)(\langle x, z\rangle \in E \wedge p_{n-1}(z, y)) \tag{2}$$

That is, $p_n(x, y)$ iff there is a path of length n from vertex x to vertex y. Likewise, $p_n(x, x)$ iff there is a cycle of length n from vertex x to itself.

Define $cycle_n$ as follows:

$$cycle_n = (\exists x)(p_n(x, x))$$

To restate the lemma, a graph is two colorable iff no vertex in the graph is in an odd cycle. This can be translated directly into a set of first-order sentences:

$$\Phi = \{\neg cycle_1, \neg cycle_3, \neg cycle_5, \neg cycle_7, \ldots\}$$

Clearly, every graph that satisfies Φ is 2-colorable. Thus, $models\,(\Phi) = 2C$, so $2C \in EC_\Delta$, which is what we wanted to prove.

Definition 7 (C') *Let Σ be a vocabulary and C a class of structures on Σ, then C' is the class of structures not in C.*

Notice that we don't want to consider the complement of C, because it will include things that are not well-formed structures too.

Lemma 2 *Properties of EC:*

- $EC \Rightarrow EC_\Delta$
- *If C is EC, then C' is also EC.*

Now that we have two notions of classes of structures, elementary and extended elementary, we wish to determine what distinguishes them. We first need a compactness theorem, equivalent to the compactness theorem for propositional logic.

Definition 8 *A set Φ of formulas is* finitely satisfiable *iff every finite subset Φ' of Φ is satisfiable.*

Theorem 4 Compactness Theorem: *A set of formulas Φ is satisfiable iff it is finitely satisfiable.*

Using this theorem, we will start by showing that graph 2-colorability is not in EC, even though, as we have shown above, it is in EC_Δ.

Theorem 5 $2C \notin EC$

Please see Lecture 25 for the proof of this theorem.