COMP 409: Lecture 21

November 09, 05

1 Review

Last time we have looked at the concept of isomorphism. Let A and B be isomorphic structures, and φ be a sentence, then $A \models \varphi$ if and only if $B \models \varphi$. We denote $A \simeq B$ to say A is isomorphic to B. We have proved that $\simeq$ is reflexive, symmetric, and transitive, that is, $\simeq$ is an equivalence relation. This means that in some sense first-order logic cannot distinguish between isomorphic statements. For instance, from the standpoint of first-order logic the structures $(\mathbb{N}, <)$ and $(\mathbb{E}, <)$ are the same, where $\mathbb{E} \subsetneq \mathbb{N}$ is the set of even numbers. This is somewhat analogous to the concept of isomorphism in group theory, where sets with "the same structure" are often treated interchangeably and are called isomorphic groups.

2 Expressibility of first-order logic

Propositional can be regarded as a special case of first-order logic, where the vocabulary is the set $Prop$ and structures are from the set of truth assignments 2^{Prop}. In discussing the expressibility of various formulas in propositional logic, we defined the notion of *adequacy* to measure how well boolean-functions can be modeled; for instance, by De Morgan's laws both $\{\neg, \vee\}$ and $\{\neg, \wedge\}$ are sufficient to express any boolean function, whereas $\{\vee, \wedge\}$ is not. Today we will try to investigate the analogous notion of "expressibility" in first-order logic.

3 Main Theorem

Theorem 1 $2C \notin EC$

Proof: We prove that $2C \notin EC$ by contradiction. (In fact, we will prove a stronger result. We will show that $\overline{EC_\Delta}$ is not in EC_Δ). Suppose $2C \in EC$. Then $2C = models(\varphi)$ for some sentence φ. Define $\overline{2C}$ all graphs not in $2C$, that is, all graphs that are not 2-colorable. As mentioned in the previous lecture, $\overline{2C}$ is EC, because EC is closed under complementation. Again, by properties of EC, $\overline{2C}$ is EC_Δ, and therefore $\overline{2C} = models(\Psi)$ for some set Ψ.

Because $2C \in EC_\Delta$, $2C = models(\Phi)$ for some set Φ. Since $2C$ and $\overline{2C}$ are mutually exclusive, $models(\Phi \cap \Psi) = models(\Phi) \cap models(\Psi) = \emptyset$. Therefore, $\Phi \cap \Psi$ is unsatisfiable.

By the compactness theorem, there are finite subsets $\Phi' \subseteq \Phi$ and $\Psi' \subseteq \Psi$ such that $\Phi' \cap \Psi'$ is unsatisfiable (take the negation of the theorem). Let $|\Phi'| = j$. From the definition of Φ, we can conclude that

$$\Phi' = \{\neg cycle_{i_1}, \neg cycle_{i_2}, \ldots, \neg cycle_{i_j}\}.$$

Let A be a graph containing only an odd cycle of length $k > max\{i_1, i_2, \ldots, i_j\}$. We have:

- $A \models \Psi$, since A contains an odd cycle, and thus it is not 2-colorable.
- $A \models \Phi'$, since the length k of the odd cycle in A is greater than all odd lengths not allowed in Φ'.

So $\Phi' \cap \Psi$ is satisfiable. This contradicts the compactness theorem. Therefore, $2C$ is not EC (and $\overline{2C}$ is not EC_Δ).

Corollary 1 *$EC \neq EC_\Delta$. Also, $\overline{EC_\Delta} \notin EC_\Delta$ (here $\overline{A}$ denotes the complement of A).*

4 Next time

We have proved that EC_Δ is more powerful than EC. Next time, we will talk more about the relationship between elementary and extended elementary.