Lecture 3

August 31, 2005

1 Unique Readability

We are trying to ensure formulas can be read in a unique way. That, we are trying to ensure that our language of formulas is not ambiguous. Hence, we prove the following:

Theorem 1 (Unique Readability) *Every non-atomic formula has a unique primary connective and (well-defined) immediate subformulas.*

To prove this, we first need to recall the following definition:

Definition 1 (Prefix) *An expression α is a prefix of β if $\beta = \alpha\gamma$. α is a strict prefix if $\gamma \neq \epsilon$. α is a nonempty prefix if $\alpha \neq \epsilon$.*

Proving Unique Readability We prove this using structural induction.

Base Case: φ is an atomic proposition. Since atomic propositions are uniquely readable, the theorem follows trivially.

The inductive step follows. We must consider two cases.

1. Let φ be $(\neg\psi)$. Then φ cannot be atomic.
   - Suppose φ is also $(\neg\psi\prime)$. Then $\psi = \psi\prime$ so ψ is unique.
   - Suppose φ is also $(\theta \circ \psi\prime)$. Then θ starts with $\neg$, which is impossible, since θ starts with (
2. Let φ be $(\theta \circ \psi)$. As before, φ cannot be atomic, and cannot be $(\neg\psi')$. Suppose φ is $(\theta' \circ' \psi')$. Then either $\theta = \theta', \circ = \circ'$,and $\psi = \psi'$, or one of θ and θ' is a strict prefix of the other. Without loss of generality, assume that θ is a strict prefix of θ'. We would like to show this impossible.

If we can show that a strict prefix of a formula cannot itself be a formula, then we can finish our proof of Unique Readability.

Let's attempt to prove the following lemma directly:

Lemma 1 *A strict prefix θ of a formula φ is not a formula.*

Proof We use structural induction

Base case: φ is atomic. This is trivially true because the only strict prefix of an atomic proposition is the empty string ϵ, which is not a formula.

Inductive step:

- φ is $(\neg\psi)$.

  θ is ϵ. Trivial.

  θ is (. Trivial.

  θ is $(\neg$. Trivial.

  θ is $(\neg\psi'$ where ψ' is a strict, nonempty prefix of φ. We are stuck here! We need to distinguish a prefix of a formula from a formula.

The straightforward approach failed in this case. We thus attempt another proof, which is based on the following definition:

Definition 2 (Parenthesis Counting) *Let*

- *l(φ) = number of '(' in φ*
- *r(φ) = number of ')' in φ*

Lemma 2 *If φ a formula, then l(φ) = r(φ)*

Proof of **Lemma 2**.

- **Base case:** φ is atomic, $\mathrm{l}(\varphi) = \mathrm{r}(\varphi) = 0$
- **Inductive case:**
  - φ is $(\neg\psi)$. Then $\mathrm{l}(\varphi) = 1 + \mathrm{l}(\psi) = 1 + \mathrm{r}(\psi) = \mathrm{r}(\varphi)$
  - φ is $(\theta \circ \psi)$. Then $\mathrm{l}(\varphi) = 1 + \mathrm{l}(\theta) + \mathrm{l}(\psi) = 1 + \mathrm{r}(\theta) + \mathrm{r}(\psi) = \mathrm{r}(\varphi)$

Lemma 3 *If θ is a nonempty strict prefix of a formula φ, then l(θ)>r(φ)*

Proof of **Lemma 3**.

Base case: Trivially true.

Inductive case:

1. φ is $(\neg\psi)$
   (a) θ is '(' : trivial
   (b) θ is '$(\neg$' : trivial
   (c) θ is '$(\neg\psi'$' where ψ' is a strict prefix of ψ. Then by IH, $\mathrm{l}(\theta) = 1 + \mathrm{l}(\psi') > 1 + \mathrm{r}(\psi') > \mathrm{r}(\psi') = \mathrm{r}(\theta)$
   (d) θ is '$(\neg\psi$'. Then $\mathrm{l}(\theta) = 1 + \mathrm{l}(\psi) > 1 + \mathrm{r}(\psi) > \mathrm{r}(\psi) = \mathrm{r}(\theta)$
2. φ is $(\alpha \circ \beta)$

(a) $\theta =$ '(': trivial

(b) $\theta =$ '$(\alpha'$', where α' is a strict non-empty prefix of α: $\ell(\theta) = 1 + \ell(\alpha'_1) > 1 + r(\alpha'_1) = r(\theta)$, as desired.

(c) $\theta =$ '$(\alpha$': $\ell(\theta) = 1 + \ell(\alpha) = 1 + r(\alpha) > r(\theta)$, as desired.

(d) $\theta =$ '$(\alpha\circ$': same as the previous case.

(e) $\theta =$ '$(\alpha \circ \beta'$', where β' is a strict non-empty prefix of β: $\ell(\theta) = 1 + \ell(\alpha) + \ell(\beta') > 1 + r(\alpha) + r(\beta') > r(\theta)$, as desired.

(f) $\theta =$ '$(\alpha \circ \beta$': $\ell(\theta) = 1 + \ell(\alpha) + \ell(\beta) = 1 + r(\alpha) + r(\beta) > r(\theta)$, as desired.

From Lemma 2 and Lemma 3 follows the statement of Lemma 1. We now know that a strict prefix of a formula is not a formula. Thus we can close the proof of unique readability.

We would like to know where the primary connective of any formula φ is, but Theorem 1 does not give us that. We need the following theorem:

Theorem 2 Parsing *The primary connective of a formula φ is the leftmost connective $\circ$ such that $\varphi = \alpha \circ \beta$ and $l(\alpha) = r(\alpha) + 1$. That is, the number of '(' preceding the primary connective is precisely one plus the number of right ')' preceding it.*

Proof of theorem

1. φ is $(\neg\theta)$. Indeed, first negation is primary connective

2. φ is $(\theta \circ \psi)$.

   $l('(\theta') = 1 + l(\theta) = 1 + r(\theta)$.

   Suppose that $\theta = \theta_1 \circ' \theta_2$ such that $l('(\theta_1') = 1 + r('(\theta_1')$, and $\theta \neq \theta_1$. Without loss of generality we can assume that θ_1 is a strict nonempty prefix of θ, so l(θ_1) > r(θ_1) and therefore $l('(\theta_1') > 1 + r('(\theta_1')$, which is a contradiction.

This theorem gives us an algorithm to parse a formula.

2 The distinction between syntax and semantics

Let's assume that our syntax talks about something. There is a distinction between syntax and semantics. Tarski explains why this is important in CS.

A program is pure syntax. When a program executes, something happens. In other words, there are physical quantities behind symbols, for ex. signals. If we have propositions $p_1, p_2, q_1, \ldots$, we have a world where these get an actual value. A world is an assignment $\tau : Prop \to \{0, 1\}$, a truth assignment.

According to Tarski, the syntax is our formulas. The world is a truth assignment. The semantics is the relationship betweens formulas and the world.

This represents a duality that is important. One point of view states that a world is a subset of $Prop$, and another that a world is a truth assignment to $Prop$. We claim these point of views are the same. Suppose $P \subseteq Prop$. Define τ_p' as $Prop \rightarrow \{0,1\}$. $\tau_p'(x) = 1$ if $x \in P$ and $\tau_p'(x) = 0$ otherwise. We call $\tau_p'(x)$ a characteristic function. Suppose $\tau : Prop \rightarrow \{0,1\}$. Define $P_\tau = x : \tau(x) = 1$.

As a side note, one fact every logician should know is the origin of the word "trivial". It is commonly used today to mean "obvious". It derives its name from the three basic ways of learning in the ancient world, the tri vias (lit. "three ways"). These three disciplines, grammar, logic, and rhetoric, were considered the most basic skills that every student should have mastery of. Therefore, for something to belong to the three ways, it must be basic enough for everyone to understand.