Lecture 30

November 14, 2005

We return to propositional logic to prove the compactness theorem. Previously, we proved the completeness theorem by resolution by showing that a a finite set of clauses is unsatisfiable iff it is refutable. This previous proof depended on the finiteness of *Prop*, which raises the question, can we extend such a proof to infinite sets of clauses?

1 Compactness

Suppose *Prop* is infinite. Then there are infinitely many formulas, and we may potentially have problems refuting an infinite unsatisfiable set of formulas. In contrast, if $Prop = p_1, \ldots, p_n$, then for every $\varphi \in Form$, $\varphi : \{0,1\}^n \mapsto \{0,1\}$. There are only 2^{2^n} different Boolean formulas. So, if *Prop* is finite, we only have finitely many formulas up to equivalence, and, therefore, a set of formulas is unsatisfiable iff some finite subset is unsatisfiable. We would like the same to hold if *Prop* is infinite.

When we proved the completeness of resolution, we showed that we can apply the operator R_p to "resolve away" propositions. When *Prop* has n propositions, this guarantees that after n steps we end up either with the empty clause or with the empty set of clauses. This argument does not hold when *Prop* is infinite. Because resolution always has two antecedents, every refutation of a set H of formulas will consider only a finite subset of H. *A priori*, it is not clear that this is enough to show that H is unsatisfiable.

Definition 1 (FSAT). *Let H be a set of formulas. We say that H is finitely satisfiable (FSAT) if every finite subset of H is satisfiable.*

Theorem 1 (Compactness Theorem). *Let H be a set of formulas. Then $H \in SAT$ iff $H \in FSAT$.*

Proof The left-to-right direction of the bi-conditional is trivial. Clearly, if H is satisfiable by some truth assignment, then every finite subset is satisfiable by the same truth assignment.

To prove it in the other direction, we consider the contrapositive form of the Compactness Theorem : A set Φ is unsatisfiable iff it has an unsatisfiable finite subset.

We now proceed to prove the more difficult direction of the Compactness Theorem. Suppose we have a truth assignment $\tau : Prop \mapsto \{0, 1\}$.

Definition 2 (Theory). *Let φ be a formula. Let $Th(\tau) = \{\varphi \mid \tau \models \varphi\}$. We read "$Th(\tau)$" as "the theory of τ".*

Lemma 1. *If $\tau \models H$, then $H \subseteq Th(\tau)$.*

Definition 3 (Exhaustive Sets). *A set $H \subseteq Form$ of formulas is **exhaustive** iff for all $\varphi \in Form$, $\varphi \in H$ or $\neg\varphi \in H$.*

Notice that the definition allows both $\varphi \in H$ and $\neg\varphi \in H$ at the same time.

Take an arbitrary formula φ. Either $\tau \models \varphi$ or $\tau \not\models \varphi$. Therefore, either $\tau \models \varphi$ or $\tau \models \neg\varphi$, i.e., either $\varphi \in Th(\tau)$ or $\neg\varphi \in Th(\tau)$.

Lemma 2. *$Th(\tau)$ is exhaustive and satisfiable, and therefore also finitely satisfiable (FSAT).*

Exhaustiveness follows from the above observation, and FSAT-ness from the fact that $Th(\tau)$ has at least one satisfying assignment (namely, τ).

We now prove an important property of exhaustive sets:

Lemma 3. *If G is $FSAT$ and exhaustive, then G is satisfiable.*

Proof:

Take $p \in Prop$. Either $p \in G$ or $\neg p \in G$, because G is exhaustive. Let $\tau_G \in 2^{Prop}$ such that $\tau_G(p) = 1$ if $p \in G$ and 0 otherwise. We show that $\tau_G \models G$. (If anything satisfies G, it is τ_G.)

Claim: $\tau_G \models G$. Let us show that $\tau_G \models \varphi$ iff $\varphi \in G$.

Proof 1 (By Structural Induction)

Basis: φ is p. By construction, $\tau_G \models p$ iff $p \in G$ iff $\tau_G(p) = 1$.

Induction:

- φ is $\neg\psi$. Then $\tau_G \models \varphi$ iff $\tau_G \not\models \psi$ iff $\psi \notin G$. So, $\varphi \in G$ iff $\psi \notin G$. The right-to-left part of the bi-conditional holds by the exhaustiveness of G. (Either $\neg\psi \in G$ or $\psi \in G$. The left-to-right holds part by finite satisfiability of G. (Otherwise, $\psi, \neg\psi \in G$, which means G is not $FSAT$). Then, by the Induction Hypothesis, $\varphi \in G$ iff $\tau_G \not\models \psi$ iff $\tau_G \models \varphi$.

- φ is $\theta \wedge \psi$. Then $\tau_G \models \varphi$ iff $\tau_G \models \theta$ and $\tau_G \models \psi$. We can use the inductive hypothesis to conclude that $\varphi \in G$ iff $\theta \in G$ and $\psi \in G$:

  $\boxed{\Leftarrow}$ Suppose $\theta \wedge \psi \in G$, but one of θ or ψ isn't. Without loss of generality, we can assume that $\theta \notin G$. Then, by exhaustiveness of G, $\neg\theta \in G$. But since G is $FSAT$, this is impossible.

  $\boxed{\Rightarrow}$ Suppose $\theta \in G, \psi \in G, \theta \wedge \psi \notin G$. Then $\neg(\theta \wedge \psi) \in G$, by exhaustiveness of G. Then $\theta, \psi, \neg(\theta \wedge \psi)$ is not $FSAT$, which yields a contradiction.

Therefore, by the inductive hypothesis, $\varphi \in G$ iff $\theta \in G$ and $\psi \in G$ iff $\tau_G \models \theta$ and $\tau_G \models \psi$ iff $\tau_G \models \varphi$. $\square$

(Notice that never used the full power of $FSAT$ in the proof of this lemma, as we only considered sets with two or three formulas.)

Proof 2 (By Contradiction)

Assume $\varphi \in G$, but $\tau_G(\varphi) = 0$. (If $\varphi \notin G$ but $\tau_G(\varphi) = 1$, then $(\neg\varphi) \in G$ and $\tau_G((\neg\varphi) = 0$.) Let $M = \{\varphi, l_1, l_2, ...l_k\} \subseteq G$, where $AP(\varphi) = \{p_1, p_2, ..., p_k\}$ and

$$l_i \quad = \quad \begin{cases} \mathrm{p}_i & if p_i \in G \\ (\neg p_i) & if p_i \notin G \end{cases}$$

Claim: M is unsatisfiable, contradicting the finite satisfiability of G.

Proof. Let $\tau'_G : AP(\varphi) \to \{0.1\}$. Suppose $\tau'_G \models M$. Clearly, $\tau_G|_{AP(\varphi)} = \tau'_G$, so, using the Relevance Lemma, we get that $\tau_G(\varphi) = 1$, a contradiction. It follows that M is unsatisfiable. $\square$

Now, suppose that G is $FSAT$, we need to find $\tau_G : Prop \mapsto \{0,1\}$ s.t. $\tau_G \models G$. We know that:

1. $G \subseteq Th(\tau)$
2. $Th(\tau)$ is exhaustive
3. $FSAT + exhaustiveness \Rightarrow satisfiability$

Intuition: If we have H and we extend it to be exhaustive but still FSAT, the original H is still $FSAT$.

Strategy: Find $G \supseteq H$ such that G is $FSAT$ and exhaustive.

Lemma 4. *If H is $FSAT$ then $H \subseteq G$ where G is $FSAT$ and exhaustive.*

(Proviso: We are only considering countably large sets of propositions, and, therefore, countably large sets of formulas. The case when we have uncountably many propositions is outside of the scope of this class).

Proof. Take $\alpha_1, \alpha_2, \ldots, \alpha_n$ to be an enumeration of $Form$. Let

$$\begin{aligned} H_0 &= H \\ H_i &= \begin{cases} H_{i-1} \cup \{\alpha_i\} & \text{if } H_{i-1} \cup \{\alpha_i\} \in FSAT \\ H_{i-1} \cup \{\neg\alpha_i\} & \text{otherwise.} \end{cases} \end{aligned}$$

Claim: Either $H_{i-1} \cup \{\alpha_i\}$ or $H_{i-1} \cup \{\neg\alpha_i\}$ is $FSAT$.

Proof of the claim: Suppose the contrary, that both $H_i \cup \{\alpha_i\}$ and $H_i \cup \{(\neg\alpha_i)\}$ are not $FSAT$. Then for some finite subset X of H_{i-1} we have $X \cup \alpha_i$ is not SAT. Also, for some finite subset Y of H_{i-1} we have $Y \cup \neg\alpha_i$ is not SAT.

Now, since $X \cup Y \subseteq H_{i-1}$, which is $FSAT$, $X \cup Y$ is satisfiable. Consider some $\tau \models X \cup Y$. By the law of excluded middle, either $\tau \models \alpha_i$ (and $X \cup \alpha_i$ is SAT) or $\tau \models \neg\alpha_i$ (and $Y \cup \neg\alpha_i$ is SAT). So either $\tau \models X \cup \alpha_i$ or $\tau \models Y \cup \neg\alpha_i$ and the claim holds.□

Now, we need to show that, in the limit, our claim still holds.

We know that

$$H_0 \subseteq H_1 \subseteq H_2 \ldots$$

Let

$$G = \bigcup_{i \in \mathbb{N}} H_i$$

G is exhaustive: Consider any $\varphi \in Form$ such that $\varphi = \alpha_i$. Either $\alpha_i \in H_i$ or $\neg\alpha_i \in H_i$.

G is $FSAT$: Suppose it isn't. Then there exists a finite subset X of G such that X is not satisfiable. Let $X = \beta_1, \ldots, \beta_n$ where each β_i was either in H_0 or it was added at some later point j_i. Take $k = max(0, j_1, \ldots, j_n)$. Then $X \subseteq H_{j_n}$. But we know that each H_i is $FSAT$, so this is impossible.

Therefore, G is SAT. Since G is a superset of H, H is also SAT. □

Corollary 1. *If Φ is unsatisfiable, then Φ is refutable.*

Proof. By the Compactness Theorem (contrapositive form), Φ has a finite unsatisfiable subset Φ'. The claim follows from the completeness of resolution. □

The proof of the Compactness Theorem is somewhat controversial because of the inductive structure of the H_i's. There could be infinitely many choices. The compactness theorem uses the controversial axiom of choice which says that there is a way to make these infinitely many choices.