Lecture 33

November 23, 2005

1 Satisfiability

Recall the satisfiability problem: given a first order logic formula φ, is φ satisfiable? This problem is equivalent to determing the validity of $\neg\varphi$. From previous lectures, we can transform φ in several different ways to force it to have a relatively simple structure:

1. Eliminate $\approx$: preserves satisfiability.
2. Convert to prenex-normal form: preserves equivalence.

Next we will show how to eliminate the existential quantifier "$\exists$" while preserving satisfiability. In this way, formulas have a fairly simple syntax, i.e., $\forall$ will remain the only quantifier.

2 Skolemization

Skolemization is the process of eliminating existential quantifiers from a formula in prenex normal form. We say that a formula $\varphi_\forall$ is in *skolem normal form* if it is of the form $(\forall y_1)\ldots(\forall y_n)\theta$, where θ is quantifier-free formula. This transformation does not preserve semantics but it does preserve satisfiability: for every formula φ, there exists a universal sentence φ_U such that φ is satisfiable iff φ_U is satisfiable. Thus, skolemization is an equisatisfiable transformation. We begin with a simple case.

Theorem 1 *Let φ be a first order formula $\varphi = (\forall x)(\exists y)\theta$. Then, there exists $f(x)$ such that φ is satisfiable iff $\varphi' = (\forall x)\theta[y \mapsto f(x)]$ is satisfiable (where y is free in θ).*

Proof.
$\boxed{\Rightarrow}$: Assume $A \models (\forall x)\,(\exists y)\,\theta$. That is, for all $a \in D^A$ there exists $b \in D^A$ such that $A, \alpha[x \mapsto a, y \mapsto b] \models \theta$. Notice that selecting b depends only on the choice for a. Let f be a function that returns the choice for b, given a.

Let the structure A' be the result of adding the function f^A to A. We claim that $A' \models \forall x\,\theta[y \mapsto f(x)]$. In other words, we want to show that for all

$a \in D^A$, we have $A', \alpha[x \mapsto a] \models \theta[y \mapsto f(x)]$ where $f^A(a)$ is some b such that $A, \alpha[x \mapsto a, y \mapsto b] \models \theta$. Note that b may not be unique, and this proof depends upon the Axiom of Choice to choose a particular b.

By a simple inductive argument, it can be shown that $A', \alpha[x \mapsto a] \models \theta[y \mapsto f(x)]$ iff $A, \alpha[x \mapsto a, y \mapsto b] \models \theta$.

$\boxed{\Leftarrow}$: Suppose A is a structure such that $A, \alpha \models \forall x\, \theta[y \mapsto f(x)]$. That is, for all $a \in D^A$ we have $A, \alpha[x \mapsto a] \models \theta[y \mapsto f(x)]$. We need to show that for all $a \in D^A$, there is some $b \in D^A$ such that $A, \alpha[x \mapsto a, y \mapsto b] \models \theta$.

Take $b = f^A(a)$. We need to prove that $A, \alpha[x \mapsto a, y \mapsto f^A(a)] \models \theta$ iff $A, \alpha[x \mapsto a] \models \theta[y \mapsto f(x)]$. We can do this by structural induction on a first order formula.

Hint: use a "Little Lemma" which says that

$$\overline{\alpha[x \mapsto a, y \mapsto b]}(t) = \overline{\alpha[x \mapsto a]}(t[y \mapsto b])$$

(Recall that $\overline{\alpha}$ is the $Term \to D$ mapping.)

$\square$

This transformation can be easily generalized. Suppose $\varphi = (\forall x_1) \dots (\forall x_n)(\exists y)\theta$ and let A be a structure such that $A \models \varphi$. Then we know that for every n-tuple $\langle a_1, \dots, a_n \rangle \in (D^A)^n$ there exists a $b \in D^A$ such that $A \models \theta[\alpha]$ where the assignment α maps x_i to a_i and y to b. This allows us to construct a table that for every n-tuple indicates the value of y that satisfies θ. In cases where y could have more than one value, we simply choose one of them.

Observe that this table maps each n-tuple in D^A to a unique element of D^A. Therefore, it defines a function $f^A(n)$ on the domain of A. We now define a new structure A' which assoiciates the unique symbol f to the function defined above. Then our transformation replaces every occurrence of y with $f(x_1, \dots, x_n)$.

3 Herbrand Universes and Structures

If φ is satisfiable, how can we find a structure A that satisfies φ? We will use the syntax as our domain.

Consider a vocabulary which contains at least one constant symbol (and without equality).

Definition 1 (Ground Term) *A* ground term *is a term containing no variables.*

We can think of a ground term as an "partially evaluated" term in the sense that the variable will be replaced with an object in the domain.

Definition 2 (Herbrand Universe) *The* Herbrand Universe *(denoted H) of a language is the set of ground terms in a language.*

Definition 3 (Herbrand Structure) *A* Herbrand Structure *of a formula φ is a structure $\mathcal{H}$ such that $D^{\mathcal{H}}$ is the Herbrand Universe of φ and $f^{\mathcal{H}}([t_1] \dots [t_n]) = [f(t_1 \dots t_n)]$.*

I.e.,

$$\mathcal{H} = (D^{\mathcal{H}}, P_1^{\mathcal{H}} \dots P_k^{\mathcal{H}}, f_1^{\mathcal{H}} \dots f_\ell^{\mathcal{H}})$$

where $D^{\mathcal{H}}$ is an *Herbrand Universe*. Terms are now both syntax and semantics: $f_i^{\mathcal{H}}(``t_1", \dots, ``t_n") = ``f_i(t_1, \dots, t_n)"$.

Definition 4 (Herbrand Model) *An* Herbrand Model $\mathcal{H}$ *of a sentence* φ *is an Herbrand structure that satisfies* φ*.*

Theorem 2 (Herbrand's Theorem) *A universal sentence* φ *without identity is satisfiable iff it is satisfiable in a Herbrand model. Alternatively: If* φ *is a satisfiable sentence, then it has an Herbrand model.*

Conclusion Every Herbrand model is countable. In particular, if it is infinite it is countably infinite.

Corollary 1 (Löwenheim-Skolem Theorem) *A first order sentence* φ *is satisfiable iff it is satisfiable in a countable structure. Alternatively: If* φ *is satisfiable, then it has a countable model.*

Corollary 2 *To check if* φ *is valid, we only have to consider countable structures.*

Corollary 3 *There is no first order sentence to express the notion "There are uncountably many elements."*

4 Validity is Recursively Enumerable

Definition 5 *Let* φ *be a universal sentence* $\forall x_1 \dots \forall x_n \theta$*. Then,*

$$ground(\varphi) = \{\theta[x_1 \mapsto t_1, \dots, x_n \mapsto t_n] \,:\, t_i \in H\}.$$

Note that elements of $ground(\varphi)$ have no variables.

Lemma 1 *Let* A *be an Herbrand structure. Then* $A \models \varphi$ *iff* $A \models ground(\varphi)$*.*

Proof. $A \models \varphi$ iff for all $t_1, \dots, t_k \in H$ we have $A, [x_1 \mapsto t_1, \dots, x_k \mapsto t_k] \models \varphi$. By a simple inductive argument on the structure of φ we can show that $A, [x_1 \mapsto t_1, \dots, x_k \mapsto t_k] \models \varphi$ iff $A \models \varphi[x_1 \mapsto t_1, \dots, x_n \mapsto t_n]$. □

Definition 6 *For every* k*-ary relation symbol* P *and* $t_1, \dots, t_k \in H$*, we introduce a proposition* $P_{t_1,\dots,t_k}$*.*

This gives us the following result:

Lemma 2 $ground(\varphi)$ *is satisfiable in an Herbrand structure iff* $ground(\varphi)[p(t_1, \dots, t_k) \mapsto p_{t_1,\dots,t_k}$ *is satisfiable.*

For example, $p(f(c)) \wedge \neg p(g(d))$ becomes $P_{f(c)} \wedge \neg P_{g(d)}$.

Thus, φ is unsatisfiable iff $ground(\varphi)$ is unsatisfiable. It follows from the Compactness Theorem for first-order logic, that some finite subset of $ground(\varphi)$ is unsatisfiable. In this way, we have reduced a formula in first order logic to one in propositional logic. In conclusion, to determine the unsatisfiability of φ, we enumerate all finite subsets of $ground(\varphi)$ and check for unsatisfiability. Finally, recall that φ is valid iff $\neg\varphi$ is unsatisfiable.

Theorem 3 (Gödel's Completeness Theorem) *Validity for first order logic is recursively enumerable.*