Lecture 4

September 2, 2005

1 Semantics

The lecture started with stating that there are two aspects (or levels) of a given language (or logic) - *syntax* and *semantics*. While *syntax* deals with the form or structure of the language, it is *semantics* that adds meaning to the form.

The *semantical domain* (also know as *domain of discourse*) is the set 2^{Prop} where *Prop* is the set of all atomic propositions. *Semantics* is the relation between $Form$, the set of all formulae, and 2^{Prop}. The set that the formula maps to tells which propositions are true and, by extension, whether a given formula is true. We call this relation *satisfaction*, represented by $\models$, where $\models \subseteq 2^{Prop} \times Form$.

In the previous two lectures we focused on the issue of logical syntax and proved the Unique Readability Theorem. This lecture describes three different views of semantics and shows that they are equivalent. In propositional logic there are only two values (truth values) *true* and *false*, or *1* and *0*. Formulas represent the syntax. What are the structures to which these formulas refer?

Definition 1 (Truth Assignment) *A truth assignment, τ, is an element of 2^{Prop}.*

There are two ways to think of truth assignments:

1. 2^{Prop} can be thought of as the power set of all propositions. A truth assignment X is an element of it, that is, $X \subseteq Prop$. Accordingly, a proposition is true if and only if it is in X

2. Or, we can think of 2^{Prop} as the set of all functions from *Prop* to the set $\{0,1\}$. Then, a truth assignment is a map $\tau : Prop \mapsto \{0,1\}$. A proposition is true if it maps to 1

These notions are equivalent since given $X \subseteq Prop$, we can define the corresponding $\tau_X : Prop \mapsto \{0,1\}$ by

$$\tau_X(p) = \begin{cases} 1 & \text{if } p \in X \\ 0 & \text{if } p \notin X \end{cases}$$

Here τ_X is the characteristic function of X. Conversely, given $\tau : Prop \mapsto \{0,1\}$, we can define the corresponding X_τ as

$$X_\tau = \{p \in Prop : \tau(p) = 1\}.$$

Next we consider three different perspectives of semantics:

1. Philosopher's view
2. Electrical engineer's view
3. Software engineer's view

2 Philosopher's view

For a philosopher, semantics is a binary relation $\models$ between truth assignments and formulas. $\tau \models \varphi$ means:

1. τ satisfies φ, or
2. τ is true of φ, or
3. φ holds at τ, or
4. τ is a model of φ.

Definition 2 ($\models$) *The relation $\models \subseteq (2^{Prop} \times Form)$ is a binary relation, where the left side is a truth assignment and the right side is a formula. $\models$ is called the* satisfaction relation, semantical relation, *or the* truth relation. *We define it inductively:*

- $\tau \models p$ for some proposition p if $\tau(p) = 1$, meaning that p holds if p is true
- $\tau \models (\neg\varphi)$ if it is not the case that $\tau \models \varphi$, that is, if $\tau \not\models \varphi$ (Note: this is true only in a 2-valued world)
- $\tau \models (\varphi \vee \psi)$ if $\tau \models \varphi$ or $\tau \models \psi$ (Remember that $\vee$ is non-exclusive)
- $\tau \models (\varphi \wedge \psi)$ if $\tau \models \varphi$ and $\tau \models \psi$
- $\tau \models (\varphi \rightarrow \psi)$ if $\tau \models \varphi$ implies $\tau \models \psi$, i.e. if $\tau \not\models \varphi$ or $\tau \models \psi$
- $\tau \models (\varphi \leftrightarrow \psi)$ if $\tau \models \varphi$ iff $\tau \models \psi$

It is important to realize that the semantics is well defined because of the Unique Readability Theorem. Consider for example the clauses:

- $\tau \models (\varphi_1 \wedge \varphi_2)$ iff $\tau \models \varphi_1$ and $\tau \models \varphi_2$
- $\tau \models (\varphi_1 \vee \varphi_2)$ iff $\tau \models \varphi_1$ or $\tau \models \varphi_2$

These clauses rely on the fact that, given a formula, we can talk about its primary connective and its immediate subformula(s) without ambiguity. Without unique readability, we would not have well-defined semantics. Suppose for example that we do not require parentheses, and consider the "formula" $p \wedge q \vee r$.

Let τ be a truth assignment such that $\tau(p) = 0$, $\tau(q) = 0$, and $\tau(r) = 1$. Since the example formula does not have a unique primary connective, its meaning may be defined in the following two ways which have 2 different values depending on where you want to put the parenthesis:

- If we do $\tau \models ((p \wedge q) \vee r)$, it is easily checked, under the truth assignment of τ, that the formula is satisfied.
- If we do $\tau \models (p \wedge (q \vee r))$, the formula is not satisfied by τ.

According to the above example, a formula could have two different truth values if it did not have a unique primary connective.

A similar kind of ambiguity exists also in natural languages. Consider the following English phrase: "John hates Jim and he likes Jill". It is not clear to whom "he" refers to–John or Jim. It is ambiguous.

3 Electrical Engineer's view

To an electrical engineer the truth assignment is simply a mapping of voltages in a wire: $\tau : Prop \rightarrow \{0, 1\}$

Operations are carried out by gates, which represent logical connectives. The value of the voltage along the wire is continuous, hence a zero value is assigned to a range of voltages and a one is assigned to a different non-overlapping range of voltages. An AND gate takes in two voltages and returns a voltage representing 1 if and only if both input voltages represented a 1. In general, a k-ary gate g is a mapping $g : \{0, 1\}^k \rightarrow \{0, 1\}$. It takes in voltages and returns a voltage as dictated by its logical specification. Here are some examples:

- $\neg : \{0, 1\} \rightarrow \{0, 1\}$
  - $\neg(0) = 1$ and $\neg(1) = 0$
- $\wedge : \{0, 1\}^2 \rightarrow \{0, 1\}$
  - $\wedge(0, 0) = \wedge(1, 0) = \wedge(0, 1) = 0$ and $\wedge(1, 1) = 1$
- $\vee : \{0, 1\}^2 \rightarrow \{0, 1\}$
  - $\vee(1, 1) = \vee(1, 0) = \vee(0, 1) = 1$ and $\vee(0, 0) = 0$
- $\rightarrow : \{0, 1\}^2 \rightarrow \{0, 1\}$
  - $\rightarrow (1, 1) = \rightarrow (0, 0) = \rightarrow (0, 1) = 1$ and $\rightarrow (1, 0) = 0$
- $\leftrightarrow : \{0, 1\}^2 \rightarrow \{0, 1\}$

– $\leftrightarrow(1,1) = \leftrightarrow(0,0) = 1$ and $\leftrightarrow(1,0) = \leftrightarrow(0,1) = 0$

We can now think of a formula as a circuit, which maps truth assignments to Boolean values: $\varphi : 2^{Prop} \to \{0,1\}$.

The logical operands, defined previously in the section on logic in philosophy, can be alternatively defined as gates that take in the inputs φ and ψ and returns the specified output.

φ	ψ	$(\neg\varphi)$	$(\varphi \wedge \psi)$	$(\varphi \vee \psi)$	$(\varphi \to \psi)$	$(\varphi \leftrightarrow \psi)$	
0	0	1	1	0	1	1	$\star$ the $\neg$ takes a single
1	0	0	1	0	0	0	immediate subformula
0	1	$\star$	1	0	1	0	and has only two
1	1	$\star$	0	1	1	1	possible inputs.

Definition 3 *Let $p \in Prop$ be a proposition, and $\tau \in 2^{Prop}$ be a truth assignment. Then the semantics is defined according to the following rules:*

- $p(\tau) = \tau(p)$ *(recall that the value of p is just voltage in a wire)*
- $(\neg\varphi)(\tau) = \neg(\varphi(\tau))$
- $(\varphi \circ \psi)(\tau) = \circ(\varphi(\tau), \psi(\tau))$

The first rule says that τ maps a proposition to its truth value. The second and third rule say that with a given τ the value of a formula can be determined from the truth values of its immediate subformulae by applying the primary connective, its functionality defined in the above table, to the values of the immediate subformulas under the same τ.

Next we prove that these two semantic formulations are equivalent. The transition between each step in the proof uses equivalence, meaning that the steps can be reversed to prove the lemma in the other direction. Notice the almost mechanical pattern of the proof. In each case the philosopher's definition is applied first, followed by the inductive hypothesis, followed by application of the above table, followed by application of the electrical engineer's definition of the connective.

Lemma 1 *Let $\varphi \in Form$ and $\tau \in 2^{Prop}$, then $\tau \models \varphi$ iff $\varphi(\tau) = 1$.*

Proof: Proof by structural induction:

- Base: $\tau \models \varphi$ iff $\tau(p) = 1$ iff $p(\tau) = 1$
- Induction:
  1. $\neg$: $\tau \models (\neg\varphi)$ iff $\tau \not\models \varphi$ iff $\varphi(\tau) = 0$ iff $\neg(\varphi(\tau)) = 1$ iff $(\neg\varphi)(\tau) = 1$
  2. $\wedge$: $\tau \models (\varphi \wedge \psi)$ iff $(\tau \models \varphi) \wedge (\tau \models \psi)$ iff $\varphi(\tau) = 1$ and $\psi(\tau) = 1$ iff $\wedge(\varphi(\tau), \psi(\tau))$ iff $(\varphi \wedge \psi)(\tau) = 1$
  3. $\vee$: the same as for 2.
  4. $\to$: $\tau \models (\varphi \to \psi)$ iff $(\tau \not\models \varphi)$ or $(\tau \models \psi)$ iff $\varphi(\tau) = 0$ or $\psi(\tau) = 1$ iff $\to(\varphi(\tau), \psi(\tau))$ iff $(\varphi \to \psi)(\tau) = 1$

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