Lecture 7

Monday, September 12^{th}, 2005

1 Introduction

The work in the previous lectures revolved around the *semantics* of propositional logic, i.e. the relationship between formulas and truth values. These results are summarized in two points:

- We defined a *satisfaction relation* $\models : \subseteq 2^{PROP} \times \textsc{Form}$ such that

$$\tau \models \varphi \Leftrightarrow \varphi(\tau) = 1 \Leftrightarrow \tau \in models(\varphi)$$

- The $\models$ relation can be computed using only a logarithmic amount of space in the size of the formula. That is,

$$\models \;\; \in \texttt{LOGSPACE}$$

2 Logical Extremes

In the set of all formulas there are two extreme types of formulas: those that are always true, and those that are always false. More formally, if $\varphi \in \textsc{Form}$ then φ can be put in one of three categories:

- φ is *valid* or a *tautology* if for every $\tau \in 2^{AP(\varphi)}$, $\tau \models \varphi$.
- φ is *unsatisfiable* or a *contradiction* or for every $\tau \in 2^{AP(\varphi)}$, $\tau \not\models \varphi$.
- φ is *satisfiable* if there is a $\tau \in 2^{AP(\varphi)}$ such that $\tau \models \varphi$.

For example, $((a \wedge (a \to b)) \to b)$ is a tautology representing the "Barbara" syllogism, which states that if Socrates is a man and all men are mortal, then Socrates is mortal.[1] $(a \wedge \neg a)$ is a contradiction, since there can be no truth assignment in which a is true and false at the same time. Finally, $(a \wedge b)$ is satisfiable, but not a tautology, since it is true if a and b are true, but false otherwise. See also the Venn diagram in Figure 1.

Which elements in this classification are the most 'interesting' depends on the point of view:

[1] The name bArbArA is a mnemonic. It represents the fact that there are three affirmative statements in the syllogism.

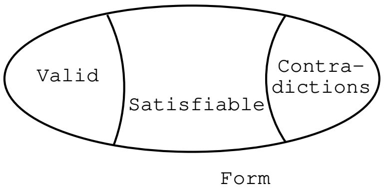

Figure 1: Venn diagram representing the classification of elements in FORM.

- Philosophers have been mostly interested in tautologies and contradictions. These are the things that are always true and never true. They represent universal truths in some sense.
- For electrical engineers, tautologies and contradictions are of little interest because the corresponding circuits have a constant output value, i.e. they are 'stuck' either at 0 or at 1. Engineers are generally interested in formulas that are satisfiable, but not valid.
- For software engineers, valid means $models(\varphi) = 2^{Prop}$, and contradiction means $models(\varphi) = \emptyset$. They, too, are mostly interested in formulas that are satisfiable, but not valid.

There is an intuitive relationship between the different classes:

Lemma 1

1. *ψ is valid $\iff$ $(\neg\psi)$ is not satisfiable.*
2. *ψ is not valid $\iff$ $(\neg\psi)$ is satisfiable.*
3. *ψ is satisfiable $\iff$ $(\neg\psi)$ is not valid.*

These can be easily proved simply from the definitions. We can now begin classifying formulas according to the above definitions. Given φ, we would like to know

- is φ satisfiable?
- is φ valid?

These questions are related as seen in the above lemmas and are fundamental to logic.

3 Logical implication

In the previous lectures, the satisfaction relation $\models$ was defined over the set $2^{Prop} \times \textsc{Form}$, i.e. it was restricted to a *single* truth assignment and a *single* formula. Now we extend this definition to sets.

We will write $T \models \Phi$ where T is a set of truth assignments and Φ is a set of formulas. That is $T \subseteq 2^{Prop}$ and $\Phi \subseteq \textsc{Form}$

3.1 Universal Lifting

To define the semantics of $\models$ over sets we use a standard construction called *universal lifting.* If applied to a relation $R \subseteq A \times B$, this construction defines another relation $R' \subseteq 2^A \times 2^B$ such that

$$R'(P, Q) \iff \forall p \in P, q \in Q : R(p, q),$$

i.e. two sets are related in R' iff each pair of elements is related in R.

3.2 Lifting $\models$

When applied to $\models$, the above construction gives us the following

Definition 1 *For a set $T \in 2^{2^{Prop}}$ and a set $\Phi \in 2^{\textsc{Form}}$, $T \models \Phi$ if for every $\tau \in T$ and every $\varphi \in \Phi$, $\tau \models \varphi$.*

This says that a set truth assignments satisfies a set of formulas if all assignments in the set satisfy all formulas.

3.3 Lifting *models*

Recall that every formula φ is associated with a set of truth assignments $models(\varphi) \subseteq 2^{Prop}$ such that $\tau \models \varphi$ iff $\tau \in models(\varphi)$. We can extend this definition to sets as well. We define *models* to a be function that takes sets of formulas to sets of truth assignments, that is $models\ :\ 2^{\textsc{Form}} \to 2^{2^{Prop}}$

We must first decide what it means for a truth assignment τ to satisfy *a set of* formulas Φ. One possibility is to interpret $\Phi = \{\varphi_1, \varphi_2, \ldots\}$ as a big conjunction $\varphi_1 \wedge \varphi_2 \wedge \ldots$ of its elements; in this case, τ satisfies Φ only if it satisfies *all* the φ_i. This is called the *conjunctive interpretation* of commas. While there are other possibilities (e.g. a big disjunction), we choose the conjunctive interpretation and arrive at the following

Definition 2 *For $\Phi \in 2^{\textsc{Form}}$,*

$$models(\Phi) := \bigcap_{\varphi \in \Phi} models(\varphi)$$

An interesting point here is that intersection over the empty set yields the universal set. This is because there are no sets to intersect, so nothing is every removed by the intersection. Formally, we have $models(\emptyset) = 2^{Prop}$

Using the above definitions we state the following lemma:

Lemma 2 *For $T \in 2^{2^{Prop}}$ and $\Phi \in 2^{\text{FORM}}$,*

$$T \models \Phi \textit{ iff } T \subseteq models(\Phi)$$

Note that the membership operator ($\in$) from the single-element definition of $\models$ has been replaced with a subset operator ($\subseteq$).

3.4 Overloading $\models$

Observe that a set of formulas also denotes a set of truth assignments. If $T = models(\varphi) \subseteq 2^{Prop}$, then we have a set of propositions which is equivalent to a truth assignment.

We previously associated sets of truth assignments with sets of formulas. We now overload $\models$ yet another time and define it as a binary relation on 2^{FORM}, which is called *logical implication*:

Definition 3 *For $\Phi, \Psi \in 2^{FORM}$:*

$$\Phi \models \Psi \textit{ if } models(\Phi) \subseteq models(\Psi)$$

By notational convention, when Ψ and Φ are singleton sets (i.e. contain a single formula $\varphi \in \Phi$ and $\psi \in \Psi$) we write $\varphi \models \psi$ instead of $\{\varphi\} \models \{\psi\}$. Note that the set of models of the left set of formulas may be equal to the set of models of the right set of formulas. For example, this is true for $\{p, q\} \models \{(p \wedge q)\}$. Every truth assignment that satisfies the formulas p and q also satisfies $(p \wedge q)$ and vice versa. However, there are also cases where $models(\Phi)$ is a strict subset of $models(\Psi)$, e.g. in the case of $\{p, q\} \models \{(p \vee q)\}$.

The name 'logical implication' already hints that there is a strong relationship between this interpretation of $\models$ and the $\rightarrow$ operator. If $\Phi \models \Psi$, then Φ represents a set of 'claims', and Ψ is true whenever the claims hold – regardless of the truth assignment.

3.5 Logical Implication vs. Material Implication

Logical implication is a very fundamental idea in logic. This relation represents a relationship between formulas where if you accept one set of formulas as true then the others will also be true, regardless of the truth assignment.

It is instructive to look at the relationship between *logical implication*, as defined above, and *material implication* which is the 'implies' operator $\rightarrow$.

Recall the definition of logical implication

- $\psi \models \varphi$: if $\tau \models \psi$ then $\tau \models \varphi, \forall\ \tau \in 2^{Prop}$

Look at the case where $\psi \to \varphi$ is valid. This means $\tau \models \psi \to \tau \models \varphi$ for all $\tau \in 2^{Prop}$. But this is the same as saying if $\tau \models \psi$ then $\tau \models \varphi$. But this is just the same as logical implication!

So we see that logical implication happens exactly when material implication holds regardless of the current set of facts. This leads to the following lemma:

Lemma 3 $\varphi \models \psi \iff \varphi \to \psi$ *is valid.*

In other words, logical implication is the validity of material implication. A shorter way of stating Lemma 3 is

$$\varphi \models \psi \iff \models (\varphi \to \psi)$$