Lecture 8

September 14, 2005

1 Review

Satisfaction

So far, we have discussed three meanings of "$\models$", successively obtained by universal lifting:

1. $\models \subseteq 2^{\textsc{Prop}} \times \textsc{Form} : \tau \models \varphi$ iff $\tau \in models(\varphi)$.
2. $\models \subseteq 2^{2^{\textsc{Prop}}} \times 2^{\textsc{Form}} : T \models \Phi$ iff $T \subseteq models(\Phi)$.
3. $\models \subseteq 2^{\textsc{Form}} \times 2^{\textsc{Form}} : \Psi \models \Phi$ when $models(\Psi) \subseteq models(\Phi)$.

Another meaning of $\models$

What does it mean to say that $\emptyset \models \Psi$, where Ψ is a set of formulas? The empty set has no "type", so it could be an empty set of formulas or an empty set of assignments. Thus, $\emptyset \models \Psi$ could fall into either of the last two cases above.

- Suppose that $\emptyset$ is an element of $2^{2^{\textsc{Prop}}}$. Since the empty set is a subset of every set, in particular $\emptyset \subseteq models(\Phi)$. In this case $\emptyset \models \Psi$ is trivially true. Obviously this interpretation is not very interesting.
- The second alternative is to consider $\emptyset$ as an element of $2^{\textsc{Form}}$. By definition, the relation $\emptyset \models \Phi$ holds iff $models(\emptyset) \subseteq models(\Phi)$. But what is $models(\emptyset)$? Our definition from the last lecture tells us that $models(\emptyset) = \cap models(\varphi)$ for all $\varphi \in \emptyset$. By convention, intersection builds from the top down, so $models(\emptyset) = 2^{2^{\textsc{Prop}}}$. In other words, $\emptyset \models \Psi$ holds iff $models(\Phi) = 2^{2^{\textsc{Prop}}}$, which is another way of saying that Φ is valid. Thus, $\emptyset \models \Phi$ means that Φ is valid.

The second interpretation is the one that is customarily used. This gives us another meaning for $\models$. Moreover, we can drop the $\emptyset$ from the left-hand side, thereby making $\models$ into an unary operator. We formalize this idea with the next definition.

Definition 1 ("$\models$" as validity) *$\models \Psi$ means that $models(\Psi) = 2^{\textsc{Prop}}$, or, in other words, that Ψ is* valid.

The overloading of $\models$ is an example of polymorphism. The types of its arguments determine the meaning of the operator.

Logical Implication

The $\models$ relationship has two meanings depending on the context. For $\tau \models \psi$ we are making a statement on membership, or satisfaction. For $\Phi \models \Psi$ we are making a statement on containment, or logical implication. In logical implication, the conclusion logically follows from the premise, whereas with material implication, if the premise is false, anything is possible. Recalling that $\rightarrow$ was a material implication, we are interested in finding connection between the two. The following lemma answers that question.

Lemma 1 ($\models$ and $\rightarrow$) *Let φ and ψ be formulas. Then $\varphi \models \psi$ iff $\models (\varphi \rightarrow \psi)$ (i.e. $(\varphi \rightarrow \psi)$ is valid). Logical implication is the validity of material implication.*

Proof:

$\boxed{\Rightarrow}$ Suppose that $\varphi \models \psi$, and let $\tau \in 2^{\textsc{Prop}}$.

- If $\tau \not\models \varphi$ then $\tau \models (\varphi \rightarrow \psi)$.
- If $\tau \models \varphi$ then $\tau \models \psi$, so $\tau \models (\varphi \rightarrow \psi)$.

$\boxed{\Leftarrow}$ Suppose that $(\varphi \rightarrow \psi)$ is valid, and again let $\tau \in 2^{\textsc{Prop}}$. To show that $\varphi \models \psi$ we need to show that $models(\varphi) \subseteq models(\psi)$. To that end, suppose that $\tau \in models(\varphi)$. Since $(\varphi \rightarrow \psi)$ is valid, every assignment (and in particular τ) is a model for it. Thus, $\tau \models (\varphi \rightarrow \psi)$, which by definition means that $\tau \models \psi$ or $\tau \not\models \varphi$. The first option is impossible (we assumed already that $\tau \in models(\varphi)$), therefore $\tau \models \psi$.

∎

If we let $\varphi \in \textsc{Form}$ and consider $\emptyset \models \varphi$, again we have two ways of interpreting the latter depending on the "type" of the empty set.

- We can think of $\emptyset$ as the empty set of truth assignment, in which case the meaning is $\emptyset \subseteq models(\varphi)$, which surely holds.
- We can think of $\emptyset$ as the empty set of formulas, in which case the meaning is $models(\emptyset) \subseteq models(\varphi)$. By definition,

$$models(\emptyset) = \bigcap_{\varphi \in \emptyset} models(\varphi) = 2^{\textsc{Prop}},$$

  so $\emptyset \models \varphi$ means that φ is *valid*.

The second interpretation is the one that is most commonly used. By convention we can also drop the $\emptyset$ from the left-hand side, just like we did in the case of $\emptyset \models \Phi$. Thus, $\models$ is a statement about validity and $\models \varphi$ means φ is valid.

2 Properties of $\models$

Monotonicity

Definition 2 *Monotonicity*

- *A function f is monotone if $x \leq y \Rightarrow f(x) \leq f(y)$.*
- *A function f is anti-monotone if $x \leq y \Rightarrow f(y) \leq f(x)$*

When we are dealing with relations we also distinguish between *left* and *right* monotonicity, according to the following definitions:

- Left Monotonicity: $xRy \wedge x' \geq x \Rightarrow x'Ry$
- Right Monotonicity: $xRy \wedge y' \geq y \Rightarrow xRy'$
- Left Anti-Monotonicity: $xRy \wedge x' \leq x \Rightarrow x'Ry$
- Right Anti-Monotonicity: $xRy \wedge y' \leq y \Rightarrow xRy'$

For example, the algebraic operator "$\leq$" is left anti-monotonic and right monotonic, while the operator "$\geq$" is left monotonic and right anti-monotonic.

To determine the monotonicity of $\models$ we first need to consider the monotonicity of "$\subseteq$". Intuitively, if $x \subseteq y$, we can remove items from x and add items to y without disrupting the validity of the statement. This means that $\subseteq$ is left anti-monotonic and right monotonic

Similar observations can be made for $models(\Phi) = \cap models(\varphi)$ for all $\varphi \in \Phi$. If $\Phi \subseteq \Phi'$ then $models(\Phi') \subseteq models(\Phi)$. The intuition is that adding more things to the set can only make the intersection smaller; more constraints make for fewer conclusions. Thus, $models$ is anti-monotone.

Now recall that $\models$ is a combination of $\subseteq$ and $models(\Phi)$. When we combine a left anti-monotonic, right monotonic relation with a anti-monotone operator, the operator reverses the monotonicity of the relation. Thus,

- $\models$ is left monotonic
- $\models$ is right anti-monotonic

Informally, this states that we can always take away consequences or add assumptions and validity will still hold. In common-sense reasoning, this is not always the case. For example, if we know that "Tweety is a bird", we infer that "Tweety can fly". But if we are now told that "Tweety is a penguin", then we withdraw the conclusion that Tweety can fly. It turns out that in day-to-day reasoning adding more assumptions doesn't necessarily preserve the logical implication. This phenomenon is called *non-monotonic reasoning* and it is an important research area in Logic.

Reflexivity, Symmetry, and Transitivity

There are three fundamental properties that each relation is tested for: *reflexivity, symmetry*, and *transitivity*. We will examine these three properties for $\models$.

1. Reflexivity: $\models$ is reflexive, because $models(\Phi) \subseteq models(\Phi)$.
2. Symmetry: $\models$ is not symmetric. To see why this is the case, consider $\Phi = \{p \wedge q\}$ and $\Psi = \{p\}$. Clearly $\Phi \models \Psi$ but the reverse is not true: $\Psi \not\models \Phi$. $\models$ is not antisymmetric either, since $\subseteq$ is not antisymmetric and $\models$ is defined as a $\subseteq$ relation between two sets.
3. Transitivity: $\models$ is transitive, because if $\Phi \models \Psi$ and $\Psi \models \Theta$, then $models(\Phi) \subseteq models(\Psi) \subseteq models(\Theta)$. Thus $\Phi \models \Theta$.

A relation that is reflexive, not symmetric and transitive is called *partial order* or *weak partial order*. From the above we can see that $\models$ is a weak partial order, because it is reflexive, not symmetric and transitive. Another weak partial order relation is $\subseteq$.

Logical Equivalence

When $\Phi \models \Psi$ and $\Psi \models \Phi$ we have a special case of logical equivalence. From the definition of $\models$ we can infer that $models(\Phi) = models(\Psi)$. This is written as $\Phi \models\!\!\!\!\dashv \Psi$. Intuitively, Φ and Ψ convey the same information.

Definition 3 (Logical Equivalence) *$\Phi \models\!\!\!\!\dashv \Psi$ iff $\Phi \models \Psi$ and $\Psi \models \Phi$.*

It is easy to infer that the relation $\models\!\!\!\!\dashv$ is reflexive, symmetric and transitive.

$\models\!\!\!\!\dashv$ partitions $2^{\textsc{Form}}$ into equivalence classes. There are two special equivalence classes:

- The equivalence class of $(p \vee (\neg p))$ contains all valid formulas or tautologies.
- The equivalence class of $(p \vee (\wedge p))$ contains all unsatisfiable formulas or contradictions.

Lemma 2 (Combinatorial Equivalence) *$\varphi \models\!\!\!\!\dashv \psi$ iff $\varphi(\tau) = \psi(\tau)$ for all $\tau \in 2^{\textsc{Prop}}$.*

In electrical engineering combinatorial equivalence is also known as *circuit equivalence*. This is an important property because electrical engineers have to pick the circuit in a certain equivalence class which best optimizes certain properties like speed or ease of implementation.

All tautologies (and all contradictions) are logically equivalent. Similar to $\models$ and $\rightarrow$, we can make a statement on the relationship of $\models\!\!\!\!\dashv$ and $\leftrightarrow$.

Lemma 3 ($\models\dashv$ and $\leftrightarrow$) *$\varphi \models\dashv \psi$ iff $\models (\varphi \leftrightarrow \psi)$ (or, in other words, $(\varphi \leftrightarrow \psi)$ is valid)*

This shows us that in some sense, everything is reducible to validity. Additionally, $\models\dashv$ allows us to make several statements on binary operators that we previously lacked the power to make.

Let Φ be a finite set $\Phi = \{\varphi_1, \varphi_2, \ldots, \varphi_n\}$. From now on, unless otherwise specified, we will deal with finite set of propositions only. Infinite sets of propositions are studied in different area of logic. For finite sets Φ we can identify the following properties:

Lemma 4 *Commutativity*

- $(\varphi \wedge \psi) \models\dashv (\psi \wedge \varphi)$
- $(\varphi \vee \psi) \models\dashv (\psi \vee \varphi)$

Lemma 5 *Associativity*

- $(\varphi \wedge (\psi \wedge \theta)) \models\dashv ((\varphi \wedge \psi) \wedge \theta)$
- $(\varphi \vee (\psi \vee \theta)) \models\dashv ((\varphi \vee \psi) \vee \theta)$

From these properties we can see that there is no need to write the parentheses when writing formulas containing multiple disjunctive or conjunctive connectives consecutively. So instead of writing $((\varphi \wedge \psi) \wedge \theta)$ we will from now on write it as $\varphi \wedge \psi \wedge \theta$.

We can make a similar housekeeping statement with $\models\dashv$ to ease making validity statements on formulas with multiple assumptions or conclusions.

Lemma 6 *$\{\varphi_1, \varphi_2, \ldots, \varphi_k\} \models\dashv \wedge_{i=1}^{k} \varphi_i$.*

Proof:

We need to show that $models(\{\varphi_1, \varphi_2, \ldots, \varphi_k\}) = models(\wedge_{i=1}^{k}\varphi_i)$.

Suppose t $\in$ models$(\{\varphi_1, \varphi_2, \ldots, \varphi_k\}) = \cap_{i=1}^{k} models(\varphi_i)$. Then $t \in models(\varphi_i)$ for all i, so $t \models \varphi_i$, $1 \le i \le k$. Therefore $t \models \wedge_{i=1}^{k}\varphi_i$.

Conversely, if $t \models \wedge_{i=1}^{k}\varphi$ then $t \models \varphi_i$, $1 \le i \le k$. So $t \in models(\varphi_i), 1 \le i \le k$. So $t \in \cap_{i=1}^{k} models(\varphi_i) = models(\{\varphi_1, \varphi_2, \ldots, \varphi_k\})$.

∎

Thus, with assumptions $\varphi_1, \varphi_2, \ldots, \varphi_k$ and conclusions $\psi_1, \psi_2, \ldots, \psi_l$, we know that the following statements are equivalent:

- $\{\varphi_1, \varphi_2, \ldots, \varphi_k\} \models \{\psi_1, \psi_2, \ldots, \psi_l\}$
- $\wedge_{i=1}^{k}\varphi_i \models \wedge_{j=1}^{l}\psi_j$
- $\models (\wedge_{i=1}^{k}\varphi_i \rightarrow \wedge_{j=1}^{l}\psi_j)$