COMP 409
Assignment No. 2

Due date: 11:59, Sept. 28, 2022

Note:

1. The submitted assignment needs to be typeset in LaTeX.
2. **Teamwork**: You are expected to complete this assignment in pairs. By signing your name on the assignment you are asserting that the work submitted was done collaboratively, and you have adhered to Rice University's Honor Code. At the very least, this entails:
   - discussing each problem and agreeing on a sketch of the solution,
   - reviewing the written solutions for technical correctness and typographical errors,
   - doing a joint codewalk over all written programs,
   - dividing labor equally (roughly), and
   - being able to explain every answer as if it is your own answer.
3. Note on the submission the total number of hours put in by each partner separately and by the two partners together.

Propositional Semantics:

1. Implement a *truth evaluator* $\mathsf{eval}(\varphi, \tau)$, which evaluates whether a formula φ holds for a truth assignment τ. Use the evaluator to answer the following question:

   Let α be the formula:

   $$((p_1 \to (p_2 \wedge p_3)) \wedge ((\neg p_1) \to (p_3 \wedge p_4))).$$

   Let β be the formula:

   $$((p_3 \to (\neg p_6)) \wedge ((\neg p_3) \to (p_4 \to p_1))).$$

   Let γ be the formula:

   $$((\neg(p_2 \wedge p_5)) \wedge (p_2 \to p_5))$$

   Let δ be the formula:

   $$(\neg(p_3 \to p_6))$$

   Evaluate the formula ψ

   $$((\alpha \wedge (\beta \wedge \gamma)) \to \delta)$$

   under the truth assignment I_1, where $I_1(p_1) = I_1(p_3) = I_1(p_5) = 0$ and $I_1(p_2) = I_1(p_4) = I_1(p_6) = 1$, as well as under the truth assignment I_2, where $I_2(p_1) = I_2(p_3) = I_2(p_5) = 1$ and $I_2(p_2) = I_2(p_4) = I_2(p_6) = 0$.

Validity:

1. Prove the validity of the following principles:
   (a) $(((p \to q) \to p) \to p)$ - Peirce's Law
   (b) $(false \to p)$ - Ex Falso Quolibet (Here $false$ is a formula that is always false.)
   (c) $((\neg p) \vee p)$: Excluded Middle ("Tertium Non Datur")
   (d) $((p \to q) \vee p)$: Weak Excluded Middle
2. Prove *Disjunctional Propositional Completeness*: if $\models (\alpha \vee \beta)$ and $AP(\alpha) \cap AP(\beta) = \emptyset$, then $\models \alpha$ or $\models \beta$

Logical Implication:

1. A binary connective $\circ$ is *commutative* if $(\theta \circ \psi)$ is logically equivalent to $(\psi \circ \theta)$; it is *associative* if $((\varphi \circ \theta) \circ \psi)$ is logically equivalent to $(\varphi \circ (\theta \circ \psi))$.

   Analyze the binary connectives $\wedge$, $\vee$, $\to$, and $\leftrightarrow$ to see whether thet are commutative and associative. Prove your claims.
2. Prove de Morgan's Laws
   - $\models ((\neg(p \wedge q)) \leftrightarrow ((\neg p) \vee (\neg q)))$
   - $\models ((\neg(p \vee q)) \leftrightarrow ((\neg p) \wedge (\neg q)))$
3. Show that logical and material equivalence coincide, that is, $\varphi \models\mathrel{\mkern-3mu}\dashv \psi$ iff $\models (\varphi \leftrightarrow \psi)$.
4. Show that $\varphi \models \psi$ iff $(\varphi \wedge \neg\psi)$ is not satisfiable.
5. Prove that logical implication is reflexive, transitive, but not symmetric. Prove that logical equivalence is an equivalence relation.
6. Show that if φ and ψ are logically equivalent and φ is valid, then ψ is valid.
7. Which of the following two sentences implies the other?

$$(p \leftrightarrow (q \leftrightarrow r)),$$
$$((p \wedge (q \wedge r)) \vee ((\neg p) \wedge ((\neg q) \wedge (\neg r)))).$$

8. Show that $\Sigma, \alpha \models \beta$ iff $\Sigma \models (\alpha \to \beta)$ (Note: Σ, α is a shorthand for $\Sigma \cup \{\alpha\}$.)
9. Prove or refute the following statements (here Σ is a set of formulas):
   - If either $\Sigma \models \alpha$ or $\Sigma \models \beta$, then $\Sigma \models (\alpha \vee \beta)$.
   - If $\Sigma \models (\alpha \vee \beta)$, then either $\Sigma \models \alpha$ or $\Sigma \models \beta$.
10. A set Σ of formulas is *independent* if no member σ of Σ is implied by $\Sigma - \{\sigma\}$. Show that a finite set of formulas always has an equivalent independent subset, but this need not hold for an infinite set of formulas.

Substitutions:

The result of *substituting* a formula θ *for* a proposition p *in* a formula φ, denoted $\varphi[p \mapsto \theta]$, is defined as follows.

- $p[p \mapsto \theta] = \theta$
- $q[p \mapsto \theta] = q$ for $q \neq p$.
- $(\neg\varphi)[p \mapsto \theta] = (\neg\varphi[p \mapsto \theta])$
- $(\varphi \circ \psi)[p \mapsto \theta] = (\varphi[p \mapsto \theta] \circ \psi[p \mapsto \theta])$ (here $\circ$ denoted an arbitrary binary connective)

Prove the following (use induction, but carefully!):

1. If φ is valid, then $\varphi[p \mapsto \theta]$ is valid.
2. If α and β are logically equivalent, then $\varphi[p \mapsto \alpha]$ and $\varphi[p \mapsto \beta]$ are logically equivalent.

Puzzle:

Three boxes are presented to you. One contains gold, the other two are empty. Each box has imprinted on it a clue as to its contents; the clues are (Box 1) "The gold is not here", (Box 2) "The gold is not here", and (Box 3) "The gold is in Box 2". Only one message is true; the other two are false. Which box has the gold?

Formalize this puzzle and solve it using propositional logic.