COMP 409
Assignment No. 3

Due date: 11:59pm, October 12, 2022

Note:

1. The submitted assignment needs to be typeset in LaTeX.
2. **Teamwork**: You are expected to complete this assignment in teams. By signing your name on the assignment you are asserting that the work submitted was done collaboratively, and that you have adhered to Rice's Honor Code. At the very least, this entails:
   - discussing each problem and agreeing on a sketch of the solution,
   - reviewing the written solutions for technical correctness and typographical errors,
   - doing a joint codewalk over all written programs,
   - dividing labor equally (roughly), and
   - being able to explain every answer as if it is your own answer.
3. Note on the submission the total number of hours put in by each partner separately and by the two partners together.

NP-completeness:

1. The *exact cover by 3-sets problem* is as follows. You are given a finite set X and a collection C of 3-element subsets of X. You have to decide whether C has an *exact cover*, i.e., a subcollection $C' \subseteq C$ such that every element of X occurs in exactly one member of C'.

   Show that this problem can be polynomially reduced to SAT.
2. The *Interger Programming* (IP) problem is to check whether a system of linear inequalities, with integral coefficients, has an integral solution. Show that this problem is NP-hard by reducing 3-SAT to it.
3. A problem $L \subseteq \Sigma^*$ is in co-NP if there is a polynomial p and a polynomial-time algorithm A such that for all $x \in \Sigma^*$ the following holds

   $x \notin L$ iff there exists some $y \in \Sigma^{\leq p(|x|)}$ such that $A(x, y) = 1$.

   The problem L is co-NP-complete if L is in co-NP and for all L' in co-NP we have that $L' \leq_p L$.

   (a) Show that L is in co-NP iff $\Sigma^* - L$ is in NP.
   (b) Show that L is co-NP-complete iff $\Sigma^* - L$ is NP-complete.
   (c) Show that if L is co-NP complete and L is in NP, then NP=co-NP.

Satisfiability:

1. Prove that for $\varphi \in \mathsf{Form}$, $\tau \in 2^{AP(\varphi)}$, and $p \in AP(\varphi)$, we have that $\tau \models \varphi$ iff $\tau|_{AP(\varphi)-\{p\}} \models \varphi[p \mapsto \tau(p)]$.

2. Prove, using only de Morgan's laws and the distributive laws (for $\wedge$ and $\vee$) that every formula is equivalent to a formula in *conjunctive normal form* (CNF), i.e., a conjunction of disjunctions of literals. (Recall that a *literal* is an atomic proposition or a negated atomic proposition.) Show that if there is a poly-time algorithm that translate each formula to its conjunctive normal form, then $P = NP$.

3. Boole's Example 5: The following problem was posed by Boole in 1854.

    > Let the observation of a class of natural productions be supposed to have led to the following general results.
    > 1st, That in whichsoever of these productions the properties A and C are missing, the property E is found, together with one of the properties B and D, but not with both.
    > 2nd, That wherever the properties A and D are found while E is missing, the properties B and C will either both be found or both be missing.
    > 3rd, That wherever the property A is found in conjunction with either B or E, or both of them, there either the property C or the property D will be found, but not both of them. And conversely, wherever the property C or D is found singly, there the property A will be found in conjunction with either B or E, or both of them.
    >
    > Let it then be required to ascertain, first, what in any particular instance may be concluded from the ascertained presence of the property A, with reference to the properties B, C, and D; also whether any relations exist independently among the properties B, C, and D. Secondly, what may be concluded in like manner respecting the property B, and the properties A, C, and D.

    Formalize Boole's example using a modern terminology and solve it, using an implementation of the splitting method (attach the code) (focus on code elegance rather than speed).

Adequacy (see lectures notes on "Formulas and Circuits")

1. Let ttof be a ternary connective such that $\mathsf{ttof}(p_1, p_2, p_3)$ is true iff $3p_1 + 2p_2 + p_3 \geq 4$. Write a $\{\neg, \wedge, \vee\}$-formula that is logically equivalent to $\mathsf{ttof}(p_1, p_2, p_3)$.

2. Let $\mathsf{succ\text{-}carry}$ be a $3n$-ary connective such that $\mathsf{succ\text{-}carry}(\mathbf{p}, \mathbf{c}, \mathbf{q})$ is true, for $\mathbf{p} = \langle p_1, \ldots, p_n \rangle$, $\mathbf{c} = \langle c_1, \ldots, c_n \rangle$, and $\mathbf{q} = \langle q_1, \ldots, q_n \rangle$, when $\mathbf{q}$ is obtained by adding 1 to $\mathbf{p}$ (modulo 2^n) with $\mathbf{c}$ being the carry. Write a $\{\neg, \wedge, \vee\}$-formula of size $O(n)$ that is logically equivalent to $\mathsf{succ\text{-}carry}(\mathbf{p}, \mathbf{c}, \mathbf{q})$.

3. Prove that $\{\vee, \wedge\}$ is *not* adequate.

4. Let the binary connective $\downarrow$ mean "nor" (i.e., $p \downarrow q$ holds if neither p nor q hold), and let the binary connective $|$ mean "not both" (i.e., $p|q$ holds if not both p and q hold). Write a truth table for $\downarrow$ and $|$ and prove that both $\{\downarrow\}$ and $\{|\}$ are adequate.

5. Let false by a 0-ary connective such that $\tau(\mathsf{false}) = 0$ for all truth assignments τ. Show that $\{\mathsf{false}, \rightarrow\}$ is adequate.

6. Let the binary connective $+$ mean "exclusive or" (i.e., $p + q$ holds if precisely one of p and q hold). Write a truth table for $+$ and prove that $\{+\}$ is not adequate.