COMP 409
Assignment No. 6 (Optional)

Due date: April 26, 2011, 11:59pm

Note:

1. The submitted assignment needs to be typeset (I recommend LaTeX).
2. **Teamwork**: You may complete this assignment in pairs. By signing your name on the assignment you are asserting that the work submitted was done collaboratively. At the very least, this entails:
   - discussing each problem and agreeing on a sketch of the solution,
   - reviewing the written solutions for technical correctness and typographical errors,
   - doing a joint codewalk over all written programs,
   - dividing labor equally (roughly), and
   - being able to explain every answer as if it is your own answer.
3. Note on the submission the total number of hours put in by each partner separately and by the two partners together.

1. An *existential-conjunctive* formula is a formula of the form $(\exists x_1)\dots(\exists x_n)\bigwedge_{i=1}^{k}\alpha_i$, where each α_i is an atomic formula. What is the complexity of evaluating existential-conjunctive queries? (Focus on upper bounds.)

   Consider a vocabulary of one binary relation symbol $\mathbf{R}$. Let $A = (D, R)$ be a structure with $D = \{1, 2, 3\}$ and $R = \{\langle 1, 2\rangle, \langle 2, 1\rangle, \langle 1, 3\rangle, \langle 3, 1\rangle, \langle 2, 3\rangle, \langle 3, 2\rangle\}$. With each graph $G = (V, E)$ we associate a sentence φ_G as follows. Let $V = \{v_1, \dots, v_n\}$. Then φ_G is

   $$(\exists x_1)\dots(\exists x_n)\bigwedge_{(v_i,v_j)\in E} R(x_i, x_j).$$

   (Note that φ_G is an existential-conjunctive formula.) Show that $A \models \varphi_G$ iff G is 3-colorable. What can you conclude from this about the complexity of evaluating existential-conjunctive queries? (Discuss upper and lower bounds.)

2. Consider a vocabulary with $\approx$ (the equality symbol) as its only relation symbol (and no function symbols). *Existential equality formulas* are formed by starting with atomic formulas and negated atomic formulas, and then closing under conjunction, disjunction, and existential quantification (negation can only be applied to atomic formulas). Show that checking satisfiability of existential equality formulas is NP-complete.

3. Let φ be a formula with $fvars(\varphi) = \{x_1, \ldots, x_n\}$. The *existential closure* of φ, denoted $\exists^*\varphi$, is $(\exists x_1)\ldots(\exists x_n)\varphi$. The *universal closure* of φ, denoted $\forall^*\varphi$, is $(\forall x_1)\ldots(\forall x_n)\varphi$.

   Show that φ is satisfiable if and only if $\exists^*\varphi$ is satisfiable and φ is valid if and only if $\forall^*\varphi$ is valid.

4. Prove formally that for every sentence φ we can construct a universal sentence φ' (over a larger vocabulary) such that φ is satisfiable iff φ' is satisfiable (Skolem Normal Form).

5. Drinker's Principle: "In every group of people one can point to one person in the group such that if that person drinks then all the people in the group drink."

   Formulate this principle in first-order logic and prove its validity.

6. Łos Problem: A k-relation over a domain D is universal if it is equal to D^k.

   Let P and Q be transitive binary relations, such that Q is symmetric and $P \cup Q$ is universal. Show that either P or Q are universal. Formalize this as a first-order logic sentence.