Lecture 22 - FOL Expressiveness - III

1 Definability of a class of structures: EC and EC_Δ

Definition 1 (EC)**.** *A class C of structures is* elementary*, denoted EC, if $C = models(\varphi)$ for some sentence φ.*

Example 1. We give some examples of EC classes, along with the sentences that define them.

- The class of undirected graphs:

$$\varphi_1 = (\forall x)(\neg E(x,x)) \wedge (\forall x)(\forall y)(E(x,y) \leftrightarrow E(y,x))$$

- The class of triangle free undirected graphs:

$$\varphi_2 = \varphi_1 \wedge ((\forall x)(\forall y)(\forall z)(E(x,y) \wedge E(y,z) \rightarrow \neg E(x,z))$$

- The class of partial orders:

$$\begin{aligned}\varphi_3 = &((\forall x)(xRx)) \wedge ((\forall x)(\forall y)(xRy \wedge yRx \rightarrow x \approx y))\\ &\wedge ((\forall x)(\forall y)(\forall z)(xRy \wedge yRz \rightarrow xRz))\end{aligned}$$

- The class of total orders: $\varphi_4 = \varphi_3 \wedge ((\forall x)(\forall y)(xRy \vee yRx))$
- The class of groups:

$$\begin{aligned}\varphi_5 = &(\forall x)(x + 0 \approx x \wedge 0 + x \approx x) \wedge (\forall x)(\exists y)(x + y \approx 0 \wedge y + x \approx 0)\\ &\wedge (\forall x)(\forall y)(\forall z)((x + y) + z \approx x + (y + z))\end{aligned}$$

- The class of abelian groups: $\varphi_6 = \varphi_5 \wedge (\forall x)(\forall y)(x + y \approx y + x)$

We note that any mathematical structure that can be defined using a finite number of axioms (e.g., groups, rings, vector spaces, boolean algebras, topological spaces) can also be defined using a single sentence, by taking the finite conjunction of the axioms. This naturally leads us to ask whether we would gain any expressiveness by allowing infinite sets of sentences to define classes of structures.

Definition 2 (EC_Δ)**.** *A class C of structures is* extended elementary*, denoted EC_Δ, if $C = models(\Phi)$ for some set of sentences Φ.*

Note that, since a finite set of sentences can be written as a single finite conjunction, the real expressive power of the above definition comes from allowing infinite number of sentences.

Our first example of an EC_Δ class is the class of infinite structures (i.e., structures with an infinite domain), denoted C_{INF}.

Theorem 1. *The class of infinite structures, C_{INF}, is EC_Δ.*

Proof. Given a natural number n, we can define the property that there are at least n elements in the domain as a first order sentence as follows:

- at least 2: $\psi_2 = \exists x_1 \exists x_2 (x_1 \not\approx x_2)$
- at least 3: $\psi_3 = \exists x_1 \exists x_2 \exists x_3 (x_1 \not\approx x_2 \wedge x_2 \not\approx x_3 \wedge x_3 \not\approx x_1)$
- at least k: $\psi_k = \exists x_1 \cdots \exists x_k \bigwedge_{i \neq j} (x_i \not\approx x_j)$

Consider the set $\Sigma = \{\psi_n : n \in N\}$. If A is a finite structure, then $A \not\models \psi_k$ for $k > |D^A|$. So $A \notin models(\Sigma)$. If A is an infinite structure, then A has at least n elements for all $n \in N$. So $A \models \psi_n$ for all $n \in N$, and thus $A \in models(\Sigma)$. Therefore $models(\Sigma)$ is exactly the class of infinite structures, and C_{INF} is EC_Δ. □

In order to discuss our second example of a class of structures that is EC_Δ, recall the following definition of k-colorable graphs:

Definition 3 (k-colorability)**.** *$G = \{V, E\}$ is k-colorable iff there is a mapping $c : V \to \{1, \ldots, k\}$ such that $E(u, v) \to c(u) \neq c(v)$.*

In the remainder of this lecture we will use $2C$ to denote the class of 2-colorable graphs. As the following well-known graph-theoretic result shows, 2-colorability has a particularly simple characterization.

Lemma 1. *A graph is 2-colorable iff it has no cycle of odd length.*

The lemma can be proved easily by induction and we leave the proof as an exercise. We will use the lemma to construct an infinite set of sentences that define $2C$.

Theorem 2. *$2C$ is EC_Δ.*

Proof. By the lemma above, $2C$ is the class of graphs which have no cycles of odd length. So a graph G is in $2C$ iff for all $n \in N$, G has no cycle of length $2n + 1$. Suppose we can represent the property that a graph has no cycle of length k as a sentence φ_k, for $k \in N$. Then $2C$ can be defined by the set of sentences $\{\varphi_{2n+1} : n \in N\}$. We now construct such a sentence.

We first define formulas $p_n(x, y)$ recursively as follows:

$$p_1(x, y) = E(x, y) \quad (1)$$
$$p_n(x, y) = (\exists z)(p_{n-1}(x, z) \wedge E(z, y)) \quad (2)$$

That is, $p_n(x, y)$ iff there is a path of length n from vertex x to vertex y. Likewise, $p_n(x, x)$ iff there is a path of length n from vertex x to itself, that is, x lies on a cycle of length n.

Then the condition that the graph has a cycle of length n can be defined by the sentence $cycle_n$ as follows:

$$cycle_n = (\exists x)(p_n(x, x))$$

Finally, the condition that the graph has no odd cycles is defined by the set of sentences Φ, defined as:

$$\Phi = \{\neg cycle_{2n+1} : n \in N\}$$

By the lemma, we have $2C = models(\Phi)$ and $2C$ is EC_Δ □

2 Relating EC and EC_Δ

Definition 4 **($\overline{C}$).** *Let Σ be a vocabulary and C a class of structures on Σ, then $\overline{C}$ is the class of structures that are not in C.*

Notice that we don't want $\overline{C}$ to just be the unrestricted complement of C, because it will include things that are not well-formed structures too. So we take complement with respect to the class of all structures. In the rest of the lecture when we refer to the complement of C, we will mean $\overline{C}$.

We now consider the relation between C and $\overline{C}$ and between EC and EC_Δ.

Lemma 2. *Properties of EC:*

1. *If C is EC, then C is EC_Δ*
2. *If C is EC, then $\overline{C}$ is also EC.*

Proof. 1. A single sentence is also a singleton set of sentences. If $C = models(\varphi)$ then $C = models(\{\varphi\})$, so we have $EC \Rightarrow EC_\Delta$.

2. If C is EC, then $C = models(\varphi)$ for some sentence φ. Since C and $\overline{C}$ are disjoint, for any structure A, we have $A \in \overline{C}$ iff $A \notin C$ iff $A \not\models \varphi$ iff $A \models \neg\varphi$. Therefore, $\overline{C} = models(\neg\varphi)$ and $\overline{C}$ is also EC.

□

Now that we have two notions of classes of structures, elementary and extended elementary, we wish to determine what distinguishes them. We know that any structure that is EC is also EC_Δ, but is the reverse true? Are there classes that are EC_Δ but not EC? And are C_{INF} and $2C$ in EC? In order to answer such questions, we turn to a fundamental notion in first order logic: *compactness.*

Definition 5 (Finite Satisfiability)**.** *A set* Φ *of formulas is* finitely satisfiable *iff every finite subset* Φ' *of* Φ *is satisfiable.*

Theorem 3. Compactness Theorem: *A set of formulas* Φ *is satisfiable iff it is finitely satisfiable.*

We will look at this theorem in more detail in the next lecture. For now, we take it as given and use it to distinguish EC and EC_Δ by showing that C_{INF} and $2C$ are not EC.

Theorem 4. *The class of infinite structures,* C_{INF}*, is not* EC*.*

Proof. We prove the statement using contradiction. Suppose that C_{INF} is EC. Then there is a sentence φ such that $C_{\mathsf{INF}} = models(\varphi)$. Now, by Theorem 1, we have $C_{\mathsf{INF}} = models(\Sigma)$, where $\Sigma = \{\psi_n : n \in N\}$. Therefore, $models(\neg\varphi) \cap models(\Sigma) = models(\neg\varphi) \cap models(\varphi) = \emptyset$, which means the set of sentences $\Phi = \{\neg\varphi\} \cup \Sigma$ is unsatisfiable. Then, by the compactness theorem, Φ is not finitely satisfiable, and so there exists a finite subset Φ' of Φ such that Φ' is not satisfiable. Since Φ' is finite, there must exist a $k \in N$ such that $\Phi' \subseteq \{\neg\varphi\} \cup \{\psi_n : n \leq k\}$. Let A be a structure with $k+1$ elements in its domain (for a concrete example, consider the complete graph on $k+1$ vertices). Then A is not infinite, so $A \not\models \varphi$ and therefore $A \models \neg\varphi$. Also A has at least n elements for all $n \leq k$, so $A \models \psi_n$ for $n \leq k$. But then A satisfies each element in Φ' so A satisfies Φ', which contradicts the fact that Φ' is not satisfiable. Thus, C_{INF} is not EC. ☐

Corollary 1. *The class of finite structures is not* EC*.*

Theorem 5. $2C$ *is not* EC*.*

Proof. The proof is very similar to the proof for C_{INF}. We again use proof by contradiction (this is a common feature of arguments that use compactness). Suppose that $2C$ is EC. Then there is some sentence φ such that $2C = models(\varphi)$. Now, by Theorem 2, we also have $2C = models(\Phi)$, where $\Phi = \{\neg cycle_{2n+1} : n \in N\}$. Then the set $\Psi = \{\neg\varphi\} \cup \Phi$ is unsatisfiable, and by compactness, there is a finite subset Ψ' of Ψ such that Ψ' is unsatisfiable. Since Ψ' is finite, there exists $k \in N$ such that $\Psi' \subseteq \{\neg\varphi\} \cup \{\neg cycle_{2n+1} : n \leq k\}$. Then a graph which is a simple cycle of length $2k+3$ satisfies Ψ', which contradicts the unsatisfiability of Ψ'. Therefore, $2C$ is not EC. ☐

So EC_Δ is strictly more powerful than EC. Earlier we saw that EC is closed under complementation, that is, C is EC iff $\overline{C}$ is EC. What can we say about $\overline{C}$ when C is EC_Δ?

Definition 6 (co-EC_Δ)**.** *A class* C *of structures is co-*EC_Δ *if* $\overline{C}$ *is* EC_Δ*.*

Example 2. The class of finite structures is co-EC_Δ, because the class of infinite structures, C_{INF}, is EC_Δ. Similarly, the class of graphs that are not 2-colorable is co-EC_Δ, because $2C$ is EC_Δ.

P/NP
$P \overset{?}{\subset} NP$
$NP \overset{?}{\neq} co-NP$
$NP \cap co-NP \overset{?}{=} P$

Table 1: P vs NP

recursive/RE
$R \subset RE$
$RE \neq co-RE$
$RE \cap co-RE = R$

Table 2: R vs RE

We can use analogies from previously studied topics to obtain a better grasp on the relationship between EC, EC_Δ and co-EC_Δ. Let us compare EC/EC_Δ with P/NP and R/RE. We know little about the relationship between P and NP, except that $P \subseteq NP$. For example, we don't know if P equals NP, or if NP equals $co-NP$. This is depicted in Table 2. In contrast to this state of affairs, we have a much better understanding of the relationship between R and RE. Recall that RE is the set of recursively enumerable languages (also called semi-decidable or semi-computable), and R is the set of recursive languages (also called decidable or computable). A language is in RE is there is some turing machine that accepts it in finite time. A language is in co-RE if there is some turing machine that rejects it in finite time. A language is in R if there is some turing machine that accepts or rejects it in finite time. From the definitions, it is clear that $RE \cap co-RE = Recursive$. The situation is shown in Table 2.

Turning our focus back to EC and EC_Δ, we ask whether our situation is analogous to P/NP or R/RE. We recall that we already have the answer to some of the analogous questions. We know that $EC \subseteq EC_\Delta$, but the equality does not hold because there are classes in EC_Δ which are not EC. The questions that we need to answer are: Is $EC_\Delta \cap co-EC_\Delta = EC$? If $EC_\Delta \neq co-EC_\Delta$? It turns out that EC/EC_Δ relationship is analogous to R/RE and the answer to both these questions is yes.

Theorem 6. $EC_\Delta \cap co\text{-}EC_\Delta = EC$

Proof. We first prove the easy direction: Let C be EC. Then $\overline{C}$ is also EC. So $\overline{C}$ is EC_Δ, and therefore, C is co-EC_Δ. Thus C is in $EC_\Delta \cap co\text{-}EC_\Delta$.

Now we prove the harder direction: Let C be EC_Δ and C also be co-EC_Δ. This means that $\overline{C}$ is EC_Δ. Thus, there exist sets of sentences, Φ and Ψ, such that $C = models(\Phi)$ and $\overline{C} = models(\Psi)$. Since no structure belongs to both C and $\overline{C}$, therefore $\Phi \cup \Psi$ is unsatisfiable. Then, by the compactness theorem, there exist finite $\Phi' \subseteq \Phi$ and finite $\Psi' \subseteq \Psi$ such that $\Phi' \cup \Psi'$ is unsatisfiable.

EC/EC_Δ
$EC \subset EC_\Delta$
$EC_\Delta \neq co - EC_\Delta$
$EC_\Delta \cap co - EC_\Delta = EC$

Table 3: EC vs EC_Δ

Let $C' = models(\Phi')$. Since $\Phi' \subseteq \Phi$, $C \subseteq C'$ (remember that the smaller set of formulas is less restrictive, therefore it has more models). Suppose $C \neq C'$, and let $A \in C' - C$. Then $A \in C'$ and $A \in \overline{C}$. This implies that $A \in models(\Phi') \cap models(\Psi) = models(\Phi' \cup \Psi)$, which means that A satisfies $\Phi' \cup \Psi$. Thus A satisfies $\Phi' \cup \Psi'$, which contradicts the unsatisfiability of $\Phi' \cup \Psi'$.

Therefore, we must have $C = C' = models(\Phi')$. Since Φ' is finite, we can construct a single sentence φ which is the conjunction of the elements of Φ'. Then $models(\varphi) = models(\Phi') = C$, and therefore C is EC. □

Corollary 2. *$EC_\Delta \neq$ co-EC_Δ.*

Corollary 3. *If C is not EC and C is EC_Δ, then $\overline{C}$ is not EC_Δ.*

The above corollary allows us to prove that something is not EC_Δ.

Example 3. The class of finite structures is not EC_Δ. This is because C_{INF} is EC_Δ but is not EC. Similarly, the class of graphs that are not 2-colorable is not EC_Δ.

Review: Given a class of structures, C, how do we place it within the EC/EC_Δ hierarchy?

- To show C is EC: Define a single sentence (or finite set of sentences) φ and show $C = models(\varphi)$.
- To show C is EC_Δ: Define a set of sentences Φ and show $C = models(\Phi)$.
- To show C is not EC: Use compactness.
- To show C is not EC_Δ: Show that $\overline{C}$ is EC_Δ and $\overline{C}$ is not EC.

Note that this is not an exhaustive approach. For example, if both C and $\overline{C}$ are not EC_Δ, then this method will not work. The class of connected graphs is an example of such a class. We will look at this example in the next lecture, when we discuss compactness.