1 Propositional semantics

1. Implement a truth evaluator eval(\phi, \tau), which evaluates whether a formula \phi holds for a truth assignment \tau. Use the evaluator to answer the question.

The code is available upon request. The results are:

\[
\begin{align*}
\alpha(I_1) = \beta(I_1) = \gamma(I_1) = \delta(I_1) = 0 & \Rightarrow \psi(I_1) = 1 \\
\alpha(I_2) = 0; \beta(I_2) = \gamma(I_2) = \delta(I_2) = 1 & \Rightarrow \psi(I_2) = 1
\end{align*}
\]

2 Validity

2.1 Prove Validity

2.1.1 \(((p \rightarrow q) \rightarrow p) \rightarrow p\) - Peirce’s Law

Let \(\tau \in 2^{\text{Prop}}\).

Case 1: \(\tau \models p\) and \(\tau \models q \rightarrow \tau \models (p \rightarrow q)\)

\[\tau \models p \text{ and } \tau \models (p \rightarrow q) \rightarrow \tau \models ((p \rightarrow q) \rightarrow p)\]

Case 2: \(\tau \models (p \rightarrow q) \rightarrow \tau \models ((p \rightarrow q) \rightarrow p)\)

\[\tau \models ((p \rightarrow q) \rightarrow p) \rightarrow \tau \models (((p \rightarrow q) \rightarrow p) \rightarrow p)\]

2.1.2 false \rightarrow p - Ex Falso Quolibet

Let \(\tau \in 2^{\text{Prop}}\). \(\tau \not\models false \rightarrow \models (false \rightarrow p)\)
2.1.3 \((\neg p \lor p)\) - Excluded Middle

Let \(\tau \in 2^{Prop}\). If \(\tau \models p\), then \(\tau \models (p \lor (\neg p))\).
If \(\tau \not\models p\), then \(\tau \models (\neg p) \implies \tau \models (p \lor (\neg p))\).
Therefore \(\models (p \lor (\neg p))\).

2.1.4 \((p \rightarrow q) \lor p\) - Weak Excluded Middle

Let \(\tau \in 2^{Prop}\). We know that \(\tau \models ((p \rightarrow q) \lor p) \iff \tau \models (p \rightarrow q)\) or \(\tau \models p\).
Case 1: \(\tau \models p\), then \(\tau \models ((p \rightarrow q) \lor p)\).
Case 2: \(\tau \not\models p\) then \(\tau \models (p \rightarrow q)\). Therefore, \(\tau \models ((p \rightarrow q) \lor p)\).

2.2 Disjunctional Propositional Completeness

Suppose \(\alpha\) and \(\beta\) were both not valid. This would mean that there exist assignments \(\tau_1 \in 2^{AP(\alpha)}\) and \(\tau_2 \in 2^{AP(\beta)}\) s.t. \(\tau_1 \not\models \alpha\) and \(\tau_2 \not\models \beta\). Note that \(\tau_1 \cap \tau_2 = \emptyset\).

Take the assignment \(\tau \in 2^{AP(\alpha \lor \beta)}\) where \(\tau = \tau_1 \cup \tau_2\). Since, because \(AP(\alpha) \cap AP(\beta) = \emptyset\), \(\tau|_{AP(\alpha)} = \tau_1\) and \(\tau|_{AP(\beta)} = \tau_2\).

Since \(\models (\alpha \lor \beta), \tau \models (\alpha \lor \beta) \implies\) either \(\tau \models \alpha\) or \(\tau \models \beta\) since it is sufficient to look at the relevant propositions only, \(\implies\) either \(\tau|_{AP(\alpha)} \models \alpha\) or \(\tau|_{AP(\beta)} \models \beta\) \(\implies\) either \(\tau_1 \models \alpha\) or \(\tau_2 \models \beta\). We know that this is not possible. Therefore our assumption must be faulty.

Hence either \(\models \alpha\) or \(\models \beta\).

3 Logical implication

1. Analyze the binary connectives \(\land, \lor, \rightarrow, \iff\) to see whether they are commutative or associative. Prove your claims.

We will show that \(\land, \lor, \text{ and } \iff\) are commutative and associative. The basic strategy in all cases is to use the commutativity and the associativity of the set-theoretic operations \(\cap, \cup, \text{ and } \equiv\). An alternative approach is to use truth tables.

- \(\land\) is commutative. Proof: \(\text{models}(\theta \land \psi) = \text{models}(\theta) \cap \text{models}(\psi) = \text{models}(\psi) \cap \text{models}(\theta) = \text{models}(\psi \land \theta)\) (since set intersection is commutative). By definition of \(\models\) it follows that \(\models (\theta \land \psi)\).

- \(\land\) is associative. Proof: \(\text{models}((\phi \land \theta) \land \psi) = \text{models}(\phi \land \theta) \cap \text{models}(\psi) = \text{models}(\phi) \cap \text{models}(\theta) \cap \text{models}(\psi) = \text{models}(\phi \land (\theta \land \psi))\). By definition of \(\models\) it follows that \(\models ((\phi \land \theta) \land \psi)\).
• ∨ is commutative. Proof: \( \text{models}(\theta \lor \psi) = \text{models}(\psi) \cup \text{models}(\theta) = \text{models}(\psi) \cup \text{models}(\theta) = \text{models}(\psi \lor \theta) \) (since set union is commutative). By definition it follows that 
\[(\theta \lor \psi) \models (\psi \lor \theta).\]

• ∨ is associative. Proof: \( \text{models}((\phi \lor \theta) \lor \psi) = \text{models}(\phi \lor \theta) \cup \text{models}(\psi) = \text{models}(\phi) \cup \text{models}(\psi) = \text{models}(\phi \lor \psi) \) (since set union is associative). By definition it follows that 
\[((\phi \lor \theta) \lor \psi) \models (\phi \lor (\theta \lor \psi))\).

• → is NOT commutative. Proof by counterexample. Take \( \text{Prop} = \{p, q, r\} \). Notice that \( \text{models}(q \rightarrow p) = \{\emptyset, \{p\}\} \cup \text{models}(p) = \{\emptyset, \{p\}\} \cup \{\emptyset, \{p, q\}\} = \{\emptyset, \{p\}, \{p, q\}\} \). Similarly, \( \text{models}(p \rightarrow q) = \{\emptyset, \{q\}, \{p, q\}\} \). Since \( \text{models}(p \rightarrow q) \neq \text{models}(q \rightarrow p) \) then \((p \rightarrow q) \) is NOT logically equivalent to \((q \rightarrow p) \).

• → is NOT associative. Proof by counterexample (different approach than above). Take \( \text{Prop} = \{p, q, r\} \) and a world \( \tau = \emptyset \) (i.e. all propositions are false). From the EE view, we have \((p \rightarrow q) \rightarrow r) (\tau) = 0 \) and \((p \rightarrow (q \rightarrow r)) (\tau) = 1 \). Then, \((p \rightarrow q) \rightarrow (p \rightarrow (q \rightarrow r)) \) is not valid, so by definition of valid \((p \rightarrow q) \rightarrow r) \) is NOT logically equivalent to \((p \rightarrow (q \rightarrow r)) \).

• ↔ is commutative. Proof: \( \text{models}(\phi \leftrightarrow \theta) = (2^{\text{AP} - \text{models}(\theta)}) \cap (2^{\text{AP} - \text{models}(\psi)}) \cup (2^{\text{AP} - \text{models}(\psi)}) \cap (2^{\text{AP} - \text{models}(\theta)}) \) by set intersection and union commutativity. By definition of \( \models \) it follows that 
\[(\theta \leftrightarrow \psi) \models (\psi \leftrightarrow \theta)\).

• ↔ is associative. Proof: \( \text{models}(\phi \leftrightarrow \theta) \leftrightarrow \psi) = \text{models}(\phi \leftrightarrow \theta) \cap \text{models}(\psi) \cup (2^{\text{AP} - \text{models}(\theta)}) \cap (2^{\text{AP} - \text{models}(\psi)})\). Expanding \( \text{models}(\phi \leftrightarrow \theta) \) = \( \text{models}(\phi) \cap \text{models}(\theta) \) \cup (2^{\text{AP} - \text{models}(\theta)}) \cap (2^{\text{AP} - \text{models}(\psi)}) \) and regrouping using associativity from set theory, we get this is equal to \( (\text{models}(\phi) \cap \text{models}(\theta) \leftrightarrow \psi) \cup (2^{\text{AP} - \text{models}(\phi)} \cap (2^{\text{AP} - \text{models}(\theta)}) \models (\phi \leftrightarrow (\theta \leftrightarrow \psi)) \). By definition of \( \models \) it follows that 
\[((\phi \leftrightarrow \theta) \leftrightarrow \psi) \models (\phi \leftrightarrow (\theta \leftrightarrow \psi))\).

2. Prove de Morgan’s Laws

- \( \models (\neg(p \land q)) \leftrightarrow ((\neg p) \lor (\neg q)) \)
- \( \models (\neg(p \lor q)) \leftrightarrow ((\neg p) \land (\neg q)) \)

3
Let

\[ \phi = (\neg (p \land q)) \leftrightarrow ((\neg p) \lor (\neg q)) \]
\[ \psi = ((\neg (p \lor q)) \leftrightarrow ((\neg p) \land (\neg q))) \]

**Proof:**

By definition, \( \phi \) is valid iff

\[ \text{models}(\phi) = 2^{\text{Prop}} \]
\[ \text{if} \quad \text{models}(\phi) = 2^{\text{AP}(\phi)} \quad (\text{by Relevance Lemma}) \]
\[ \text{if} \quad \phi(\tau) = 1, \forall \tau \in 2^{\text{AP}(\phi)} \quad (\text{by Lemma from class}) \]

Similarly, \( \psi \) is valid iff

\[ \text{models}(\psi) = 2^{\text{Prop}} \]
\[ \text{if} \quad \text{models}(\psi) = 2^{\text{AP}(\psi)} \quad (\text{by Relevance Lemma}) \]
\[ \text{if} \quad \psi(\tau) = 1, \forall \tau \in 2^{\text{AP}(\psi)} \quad (\text{by Lemma from class}) \]

The following table shows that \( \text{models}(\phi) = 2^{\text{AP}(\phi)} \) and \( \text{models}(\psi) = 2^{\text{AP}(\psi)} \).

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>((\neg (p \land q)))</th>
<th>((\neg p) \lor (\neg q))</th>
<th>( \phi )</th>
<th>((\neg (p \lor q)))</th>
<th>((\neg p) \land (\neg q))</th>
<th>( \psi )</th>
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</tbody>
</table>

3. **Show that logical and material equivalence coincide, that is, \( \phi \models \models \psi \) iff \( \models (\phi \leftrightarrow \psi) \).**

By definition,

\[ \phi \models \models \psi \]
\[ \text{if} \quad \phi \models \psi \quad \text{and} \quad \psi \models \phi \quad (\text{by definition of \( \models \models \)}) \]
\[ \text{if} \quad \models (\phi \rightarrow \psi) \quad \text{and} \quad \models (\phi \rightarrow \psi) \quad (\text{by Lemma 1 in Lecture 8}) \]
\[ \text{if} \quad \models (\phi \leftrightarrow \psi) \quad (\text{by definition of \( \leftrightarrow \)}) \]
4. Show that $\phi \models \psi$ iff $(\phi \land \neg \psi)$ is not satisfiable.

Proof:
By definition,

$(\phi \land \neg \psi)$ is not satisfiable
iff $\neg (\phi \land \neg \psi)$ is valid
iff $\models \neg (\phi \land \neg \psi)$
iff $\models (\neg \phi \lor \psi)$ (by de Morgan’s Law)
iff $\models (\phi \rightarrow \psi)$ (by definition of $\rightarrow$)
iff $\phi \models \psi$ (by definition of logical implication)

5. Prove that $\models$ is reflexive, transitive, but not symmetric. Prove that $\models\models$ is reflexive, transitive, and symmetric.

Notice that $\models$ is not a relation between $2^{\text{Prop}}$ and $\text{Form}$, nor between $2^{\text{Prop}}$ and $2^{\text{Form}}$ (otherwise neither of the properties would make sense).
The only option that is left is $\models \subseteq 2^{\text{Form}} \times 2^{\text{Form}}$.

Let $\Phi, \Psi \in 2^{\text{Form}}$

- $\models$ is reflexive. Proof: since $\text{models}(\Phi) \subseteq \text{models}(\Phi)$ then $\Phi \models \Phi$ by definition.
- $\models$ is transitive. Proof: if $\Phi \models \Psi$ and $\Psi \models \Theta$ then $\text{models}(\Phi) \subseteq \text{models}(\Psi)$ and $\text{models}(\Psi) \subseteq \text{models}(\Theta)$. Since subset is a transitive relation, $\text{models}(\Psi) \subseteq \text{models}(\Theta)$, so $\Phi \models \Theta$.
- $\models$ is NOT symmetric. Proof by counterexample. Let $\Phi = \{(p \land q)\}$, $\Psi = \{p\}$. $\text{Prop} = \{p,q\}$. Then, $\text{models}(\Phi) = \{\{p,q\}\}$ and $\text{models}(\Psi) = \{\{p\}\}$. Since $\text{models}(\Phi) \subseteq \text{models}(\Psi)$ but $\text{models}(\Psi) \not\subseteq \text{models}(\Phi)$, it follows from the definition of $\models$ that $\Phi \models \Psi$ but $\Psi \not\models \Phi$.

In the case of $\models\models$ we also have sets of formulas on both sides.

- $\models\models$ is reflexive. Proof: Since $\text{models}(\Phi) = \text{models}(\Phi)$ then $\Phi \models\models \Phi$ by definition.
- $\models\models$ is transitive. Proof: if $\Phi \models\models \Psi$ and $\Psi \models\models \Theta$ then $\text{models}(\Phi) = \text{models}(\Psi)$ and $\text{models}(\Psi) = \text{models}(\Theta)$, so $\text{models}(\Phi) = \text{models}(\Theta)$ and by definition $\Phi \models\models \Theta$.
- $\models\models$ is symmetric. Proof: $\Phi \models\models \Psi$ iff $\text{models}(\Phi) = \text{models}(\Psi)$ iff $\text{models}(\Psi) = \text{models}(\Phi)$ iff $\Psi \models\models \Phi$.

6. Show that if $\phi$ and $\psi$ are logically equivalent and $\phi$ is valid, then $\psi$ is valid.

Restating the problem formally, we show that if $\phi \models\models \psi$ and $\models \phi$, then $\models \psi$ (by definition of logical equivalence and logical validity).
Proof: By definition of |=|=, φ |=|=ψ iff models(φ) = models(ψ). By definition of logically valid, |= φ iff models(φ) = 2^{Prop}. Thus, models(ψ) = 2^{Prop} and hence |= ψ.

7. Which of the formulas implies the other?

Let

φ = (p ↔ (q ↔ r))
ψ = ((p ∧ (q ∧ r)) ∨ ((¬p) ∧ (¬q) ∧ (¬r)))

Then, models(φ) = \{\{p\}, \{p,q,r\}, \{q\}, \{r\}\} and models(ψ) = \{\{p,q,r\}, \emptyset\}.

Since neither models(φ) ⊆ models(ψ) nor models(ψ) ⊆ models(φ) then neither φ |= ψ nor ψ |= φ.

8. Show that Σ ∪ {α} |= β iff Σ |= (α → β).

First, we need some help:

Lemma 1. If A and B are sets, then A ⊆ (A ∩ B) ∪ B.

Proof. Suppose that a ∈ A. We must show that a ∈ A ⊆ (A ∩ B) ∪ B. Either a ∈ B or a /∈ B. In the first case a ∈ (A ∩ B), in the second, a ∈ B. In either case a ∈ A ⊆ (A ∩ B) ∪ B, which is what we had to prove.

Proof of the main problem.

Σ ∪ {α} |= β
iff models(Σ ∪ {α}) ⊆ models(β)
iff models(Σ) ∩ models(α) ⊆ models(β)
iff (models(Σ) ∩ models(α)) ∪ (2^{Prop} − models(α))
⊆ (2^{Prop} − models(α)) ∪ models(β) (1)
(union the same set to each side)
iff models(Σ) ⊆ (2^{Prop} − models(α)) ∪ models(β) (by Lemma) (2)
iff models(Σ) ⊆ models(α → β) (by defn of →)
iff Σ |= (α → β) (by defn of |=)

Note that the step from (1) to (2) is justified by Lemma 1, but this is not the case when we traverse the proof in the opposite direction. For this we need another lemma, which we state and prove below.

Lemma 2. Let Σ, U, A and B be sets. If

Σ ⊆ (U − A) ∪ B, then

(Σ ∩ A) ∪ (U − A) ⊆ (U − A) ∪ B.
Proof. Since \((\Sigma \cap A) \subseteq \Sigma\) and \(\Sigma \subseteq (U - A) \cup B\), we can conclude that \((\Sigma \cap A) \subseteq (U - A) \cup B\). Clearly \((U - A) \subseteq (U - A) \cup B\), therefore the left-hand side is the union of two sets which are both contained in the right-hand side.

The proof of Lemma 2 concludes the proof of the main problem. 

9. Prove or refute the following statements.

- If either \(\Sigma \models \alpha\) or \(\Sigma \models \beta\) then \(\Sigma \models (\alpha \lor \beta)\). **TRUE.**
  Proof:
  
  If either \(\text{models}(\Sigma) \subseteq \text{models}(\alpha)\) or 
  \(\text{models}(\Sigma) \subseteq \text{models}(\beta)\) then 
  \(\text{models}(\Sigma) \subseteq \text{models}(\alpha) \cup \text{models}(\beta)\)
  \(\subseteq \text{models}(\alpha \lor \beta)\) by definition.

  Then \(\Sigma \models (\alpha \lor \beta)\).

- If \(\Sigma \models (\alpha \lor \beta)\), then either \(\Sigma \models \alpha\) or \(\Sigma \models \beta\). **FALSE.**
  Proof by counterexample:
  Take \(Prop = \{p, q\}, \Sigma = \{p\}\), \(\alpha = (p \land q)\), \(\beta = (\neg(p \land q))\) Then:
  
  \[
  \begin{align*}
  \text{models}(\Sigma) &= \{\{p\}, \{p, q\}\} \\
  \text{models}(\alpha) &= \{\{p, q\}\} \\
  \text{models}(\beta) &= \{\emptyset, \{p\}, \{q\}\}
  \end{align*}
  \]

  Here, \(\text{models}(\Sigma) \subseteq \text{models}(\alpha) \cup \text{models}(\beta)\) iff \(\Sigma \models (\alpha \lor \beta)\)
  but \(\text{models}(\Sigma) \nsubseteq \text{models}(\alpha)\) therefore \(\Sigma \not\models \alpha\)
  and \(\text{models}(\Sigma) \nsubseteq \text{models}(\beta)\) therefore \(\Sigma \not\models \beta\)

  Which completes the proof.

10. A set \(\Sigma\) of formulas is independent if \(\Sigma - \{\sigma\} \not\models \sigma, \forall \sigma \in \Sigma\).

Let \(\Sigma = \{\sigma_1, \sigma_2, ..., \sigma_n\}\) be a finite set of formulas, if \(\Sigma\) is independent already then we have the final result we wanted.

If \(\Sigma\) is not independent, then it means that there is at least one \(\sigma_i\), such that \(\sigma_1 \land \sigma_2 \land ... \land \sigma_{i-1} \land \sigma_{i+1} \land ... \land \sigma_n \models \sigma_i\). That is \(\sigma_i\) is implied by the conjunction all of the formulas from \(\Sigma\) except \(\sigma_i\) itself.

In order to make \(\Sigma\) an independent set of formulas, we can eliminate \(\sigma_i\) from the set. The set \(\Sigma - \{\sigma_i\}\) will still be equivalent to the original set. We can continue to do this until there are no more dependent formulas in
the resulting set.

On the other hand, if $\Sigma$ is an infinite set of formulas ($\Sigma = \{\sigma_1, \sigma_2, \ldots\}$), such as:

\[
\begin{align*}
\sigma_1 &= p_1 \\
\sigma_2 &= p_1 \land p_2 \\
& \vdots \\
\sigma_n &= p_1 \land p_2 \land \cdots \land p_n \\
& \vdots
\end{align*}
\]

In this case, the set itself is not independent. If we take a finite subset of the formulas, it’s not going to be equivalent, since an infinite number of formulas can not be generated from the finite subset (any formula with atomic propositions with subscripts greater than the greatest subscript present in the subset).

If we take an infinite subset, say by eliminating $\sigma_i$ from the set, we are still able to generate the missing formula from any $\sigma_j, j > i$. Take $\sigma_{i+1}$, for example, if $p_1 \land p_2 \land \cdots \land p_i \land p_{i+1}$ holds then $p_1 \land p_2 \land \cdots \land p_i$ holds also. This infinite subset is equivalent, but it is not independent since it possesses the same properties as the original set. Any other infinite subset is also equivalent to the original set, due to the same reasoning.

4 Substitutions

If $\phi$ is valid, then $\phi[p \mapsto \theta]$ is valid.

1. If $|= \varphi$, then $|= \varphi[p \mapsto \theta]$.

Proof: Suppose that $|= \varphi$. Let $\tau \in 2^{PROP}$. We wish to show that $\tau(\varphi[p \mapsto \theta]) = 1$. Let $\sigma \in 2^{PROP}$ be identical to $\tau$ except that $\sigma(p) = \tau(\theta)$. First we show that $\tau(\varphi[p \mapsto \theta]) = \sigma(\varphi)$.

Case $\varphi = p$. Then $\tau(p[p \mapsto \theta]) = \tau(\theta) = \sigma(p)$.

Case $\varphi = q \neq p$. Then $\tau(q[p \mapsto \theta]) = \tau(q) = \sigma(q)$.

Case $\varphi = (\neg \psi)$. Then $\tau((\neg \psi)[p \mapsto \theta]) = \tau((\neg(\psi[p \mapsto \theta])) = \neg(\tau(\psi[p \mapsto \theta])) = \neg(\sigma(\psi)) = \neg(\neg(\psi)) = \sigma(\neg(\psi)) = \sigma(\neg \psi)$.

Case $\varphi = (\theta \circ \psi)$. Then $\tau((\theta \circ \psi)[p \mapsto \theta]) = \tau(\theta[p \mapsto \theta] \circ \psi[p \mapsto \theta]) = \circ(\tau(\theta[p \mapsto \theta]), \tau(\psi[p \mapsto \theta])) = \circ(\sigma(\theta), \sigma(\psi)) = \sigma(\theta \circ \psi)$.

Now, since every truth assignment makes $\varphi$ true (because it is valid), $\sigma$ in particular makes it true, and so $\tau$ makes $\varphi[p \mapsto \theta]$ true.
2. If $\alpha$ is equivalent to $\beta$, then $\varphi[p \to \alpha]$ is equivalent to $\varphi[p \to \beta]$.

Proof: Let $\tau$ be an arbitrary truth assignment. We wish to show $\varphi[p \to \alpha]$ is equivalent to $\varphi[p \to \beta]$. We proceed by structural induction.

Case $\varphi = p$. Then $\tau(p[p \to \alpha]) = \tau(p) = \tau(p[p \to \beta])$.

Case $\varphi = q \neq p$. Then $\tau(q[p \to \alpha]) = \tau(q) = \tau(q[p \to \beta])$.

Case $\varphi = (\neg \theta)$. Then $\tau((\neg \theta)[p \to \alpha]) = \neg(\tau(\theta[p \to \alpha])) = \neg(\tau(\theta[p \to \beta]))$ (by the inductive hypothesis) $= \tau((\neg \theta)[p \to \beta])$.

Case $\varphi = (\theta \circ \psi)$. Then $\tau((\theta \circ \psi)[p \to \alpha]) = \tau(\theta[p \to \alpha] \circ \psi[p \to \alpha]) = \circ(\tau(\theta[p \to \alpha]), \tau(\psi[p \to \alpha])) = \circ(\tau(\theta[p \to \beta]), \tau(\psi[p \to \beta]))$ (I.H.) $= \tau((\theta \circ \psi)[p \to \beta])$.

5 Puzzle

Let the assertion “Box $i$ has the gold” be represented by $p_i$ and the assertion “The clue on Box $i$ says the truth” be represented by $q_i$. The following facts are given:

- Only one box have the gold: $og = ((p_1 \lor p_2 \lor p_3) \land (\neg(p_1 \land p_2) \land \neg(p_1 \land p_3) \land \neg(p_2 \land p_3)))$.
- Box1: $b_1 = (q_1 \leftrightarrow \neg p_1)$.
- Box2: $b_2 = (q_2 \leftrightarrow \neg p_2)$.
- Box3: $b_3 = (q_3 \leftrightarrow p_3)$.
- Only one clue is true: $ot = (q_1 \lor q_2 \lor q_3) \land (\neg(q_1 \land q_2) \land \neg(q_1 \land q_3) \land \neg(q_2 \land q_3))$.

We need a satisfying assignment to $og \land b_1 \land b_2 \land b_3 \land ot$ to solve the puzzle. The only satisfying assignment would be $p_1 = 1, p_2 = 0, p_3 = 0, q_1 = 0, q_2 = 1, q_3 = 0$, which means box 1 contains the gold.