Homework 7 Sample Solutions

1. Let $A$ be the standard model of arithmetics, with the set $N$ of natural numbers as the domain of $A$, and with the binary relation $<$, the constants 0 and 1, and the functions $+$ and $\cdot$ getting their standard interpretations in $A$.

Show that there is a structure $A'$ that is elementarily equivalent to $A$ but not isomorphic to $A$. (Hint: Start with the set of all sentences that hold in $A$ and extend them with an infinite set of sentences that forces a new constant to be greater than all natural numbers. Use compactness to show that this set of sentence is satisfiable.)

What can you conclude from this?

Let $\Sigma = \{ \varphi : A \models \varphi \}$ be the set of all sentence that are true in $A$. Note that, by definition, $A \models \Sigma$. We can then define number $i$ to be the term:

- $number_0 = 0$
- $number_i = (1 + number_{i-1})$

Notice that, in $A$, the term $number_i$ evaluates to $i$, i.e. $A(number_i) = i$. We then define a new 0-ary function symbol $n$ and define $\psi_i = (number_i < n)$ to be a formula that says (in $A$) that $i < n$. Then $(N, <, 0, 1, +, \cdot)$ satisfies $\Sigma \cup \{ \psi_i : i < k \}$, since $A = (N, <, 0, 1, +, \cdot)$ satisfies $\Sigma$ and $i < k$ for all $i < k$. It follows that $\Sigma' = \Sigma \cup \{ \psi_i : i \in N \}$ is finitely satisfiable, and hence, by compactness, $\Sigma'$ is satisfiable. Let $(N', <', 0', 1', +', \cdot')$ be a model of $\Sigma'$. Since $\Sigma \subseteq \Sigma'$, all models of $\Sigma'$ are also models of $\Sigma$; since the symbol $n$ does not occur in $\Sigma$, we get that $A' = (N', <', 0', 1', +', \cdot')$ is a model of $\Sigma$. But $\Sigma$ was defined to be the set of all sentences satisfied by $A$, and so we conclude that $A'$ is elementarily equivalent to $A$.

However, $A'$ is not isomorphic to $A$. Assume for the sake of a contradiction that $f$ is an isomorphism from $A'$ to $A$ and consider $f(n')$. Let $A'(number_i)$ denote the element of $A'$ obtained by evaluating the term $number_i$ in the structure $A'$. Then, since $(A', n')$ satisfies $\{ \psi_i : i \in N \}$, we get that $A'(number_i) <' n'$ for all $i \in N$. Since $f$ is a homomorphism, this holds if and only if $f(A'(number_i)) < f(n')$ for all $i \in N$. By definition of a homomorphism, $f(A'(number_i)) = A(number_i) = i$. Therefore $i < f(n')$ for all $i \in N$, but if $f(n') \in N$ then this implies that $f(n') < f(n')$, a contradiction. Thus $A'$ is not isomorphic to $A$.

We can conclude from this nonstandard models of arithmetics exist.
2. Prove the Skolem-Normal-Form Theorem in its full generality: Every prenexnormal-form first-order sentence is equisatisfiable with a universal sentence.

Hint: Eliminate existential quantifiers from the inside out. Formulate the induction hypothesis carefully.

We prove the following statement by induction on $i$.

**Statement:** Every prenexnormal-form first-order sentence $\varphi$ containing exactly $i$ existential quantifiers is equisatisfiable with a universal sentence.

**Basis:** If $i = 0$, then $\varphi$ is a prenexnormal-form sentence containing no existential quantifiers. Thus $\varphi$ is a universal sentence and equisatisfiable with itself.

**Inductive Step:** Let $i > 0$ and let $\varphi$ be a prenexnormal-form sentence containing exactly $i$ existential quantifiers. Then $\varphi = (Q_1 x_1) \cdots (Q_n x_n) \theta$, where $Q_k \in \{\forall, \exists\}$ and $\theta$ is a universal formula. Let $j$ be the index of the first occurrence of $\exists$, so that $Q_j = \exists$ and $Q_k = \forall$ for all $1 \leq k < j$.

We can assume without loss of generality that $Q_1 = \forall$. If instead $Q_1 = \exists$, let $x_0$ be a fresh variable and consider $(\forall x_0)\varphi$. Then $(\forall x_0)\varphi$ is equisatisfiable with $\varphi$. Moreover, $(\forall x_0)\varphi$ is a prenexnormal-form sentence containing exactly $i$ existential quantifiers, but the first quantifier of $(\forall x_0)\varphi$ is not existential. We are therefore free to assume that $j > 1$.

Then $\varphi = (\forall x_1) \cdots (\forall x_{j-1}) (\exists x_j) \psi$ for some formula $\psi$. By Theorem 4 of the Skolemization lecture notes there exists a term $f(x_1, \cdots, x_{j-1})$ such that $\varphi$ is satisfiable if $\varphi' = (\forall x_1) \cdots (\forall x_{j-1}) \psi[x_j \mapsto f(x_1, \cdots, x_{j-1})]$ is satisfiable. Since $\psi$ and $\psi[x_j \mapsto f(x_1, \cdots, x_{j-1})]$ contain an identical number of existential quantifiers, it follows that $\varphi'$ contains exactly $i - 1$ existential quantifiers. Moreover, $\varphi'$ is still a prenexnormal-form first-order sentence. By the inductive hypothesis, $\varphi'$ is equisatisfiable with a universal sentence. Since $\varphi$ is equisatisfiable with $\varphi'$, it follows that $\varphi$ is equisatisfiable with a universal sentence as well.

By induction, the statement holds for all $i$. This implies the Skolem-Normal-Form Theorem.

3. Formulate in first-order logic: “for every boy, there is a girl who loves only him”. What can you say about the number of boys and girls?

$B(b)$ means $b$ is a boy; $G(g)$ means $g$ is a girl; $L(x, y)$ means that $x$ loves $y$.

We then formalize the statement as:

$$(\forall b)(B(b) \rightarrow (\exists g)(G(g) \land (\forall b')(L(g, b') \leftrightarrow (b \approx b'))))$$

If the statement is true then there must be at least as many girls as boys.

4. Formulate and prove the validity of Drinkers Principle: in every group of people one can point to one person in the group such that if that person drinks then all the people in the group drink.
Prove validity first by direct semantical reasoning and then by appealing to Herbrand’s Theorem.

Drinks(p) means that p drinks. We then then formalize the statement as:

\[ \varphi = (\exists p)(\text{Drinks}(p) \rightarrow (\forall m)\text{Drinks}(m)) \]

We first prove validity of \( \varphi \) by direct semantical reasoning. To show \( \varphi \) is valid, we show that \( \varphi \) is satisfied under all structures \((G, \text{Drinks})\). Every structure with this vocabulary must fall into one of three cases:

- **Drinks = \emptyset**. In this case nobody in the group drinks. If this is the case then the formula is true since the first part of the implication will always be false.
- **Drinks = G**. In this case everybody in the group drinks. If this is the case the formula will also be true since the latter part of the implication is always true.
- **Drinks \subset G**. In this case the people that drink is a proper subset of the entire group. If this is the case then there is at least one person in the group that does not drink (since the relation is a proper subset). The formula will be valid because there will always exist a person for which the first part of the implication is false, and thus the formula will be true.

We next prove validity of \( \varphi \) through Herbrand’s Theorem. \( \varphi \) is valid if and only if \( \neg \varphi \) is unsatisfiable. Converting \( \neg \varphi \) to prenex normal form, we get that \( \varphi' = (\forall p)(\exists m)(\text{Drinks}(p) \land \neg \text{Drinks}(m)) \) is logically equivalent to \( \neg \varphi \). By Skolemization, \( \varphi' \) is satisfiable iff \( \varphi'' = (\forall p)(\text{Drinks}(p) \land \neg \text{Drinks}(f(p))) \) is satisfiable.

We claim that no Herbrand Structure satisfies \( \varphi'' \). Let \( \mathcal{H} \) be a Herbrand Structure of \( \varphi'' \) and let \( x \) be an element of the Herbrand Universe of \( \varphi'' \). Assume for the sake of a contradiction that \( \mathcal{H} \models \varphi'' \). By the semantics of \( \forall \), we must have \([p \rightarrow x] \models (\text{Drinks}(p) \land \neg \text{Drinks}(f(p)))\). Thus \([p \rightarrow x] \models \neg \text{Drinks}(f(p)) \) and so \( \neg \text{Drinks}(f(x)) \) must hold in \( \mathcal{H} \). On the other hand, we must also have \([p \rightarrow f(x)] \models (\text{Drinks}(p) \land \neg \text{Drinks}(f(p)))\). Thus \([p \rightarrow f(x)] \models \text{Drinks}(p) \) and so \( \text{Drinks}(f(x)) \) must hold in \( \mathcal{H} \). This is a contradiction, since exactly one of \( \neg \text{Drinks}(f(x)) \) and \( \text{Drinks}(f(x)) \) can hold in \( \mathcal{H} \).

Thus no Herbrand Structure satisfies \( \varphi'' \). By Herbrand’s Theorem, since \( \varphi'' \) is a universal sentence without identity this implies that \( \varphi'' \) is unsatisfiable. It follows that \( \neg \varphi \) is unsatisfiable and so \( \varphi \) is valid.

5. Can you formulate in first-order logic the sentence “There are some critics who admire one another”?

\( A(x, y) \) means that \( x \) admires \( y \). We then then formalize the statement as:

\[ (\exists x)(\exists y)(A(x, y) \land A(y, x)) \]
There is a related sentence that is famously not representable in first-order logic, namely “There are some critics who admire only one another”. In second-order logic, we can formalize this new statement as:

\[(\exists S)((\exists z)S(z) \land (\forall x)(\forall y)((S(x) \land A(x, y)) \to (x \neq y \land S(y))))\]

This statement cannot be formalized in first-order logic. To prove this, assume for the sake of a contradiction that \(\varphi\) encodes this statement in the vocabulary containing the single binary relation \(A\) and equality. Let \(\varphi'\) consist of \(\varphi\) with all instances of “\(A(x, y)\)” for all terms \(x\) and \(y\) replaced by “\((x = 0 \lor x = y + 1)\)”. Then \(\varphi'\) is a first-order logic sentence in the vocabulary of the standard model of arithmetics discussed in Problem 1. \(\varphi'\) expresses the sentence “There is a nonempty set \(S\) of nonzero numbers that are only successors of one another”.

Notice that \(\varphi'\) is false in the standard model of arithmetic \(\mathcal{A}\). In particular, if \(k \in S\) then since \(k\) is the successor of \(k - 1\) we must have \(k - 1 \in S\) as well. Since \(0 \notin S\), it follows that \(S\) has no smallest element; this is impossible in the standard model of arithmetic.

However, \(\varphi'\) is true in the nonstandard model of arithmetic \(\mathcal{A}'\) from Problem 1. In particular, let \(S\) be the set of all elements of \(\mathcal{A}'\) not of the form \(\mathcal{A}'(\text{number}_i)\) for some \(i \in \mathbb{N}\). \(S\) is nonempty since \(n' \in S\), and the numbers in \(S\) are not successors of any natural number and so can only be successors of one another.

Since \(\mathcal{A}\) and \(\mathcal{A}'\) cannot be distinguished in first-order logic, it follows that \(\varphi'\) cannot be represented in first-order logic. It is therefore impossible to formulate “There are some critics who admire only one another” in first-order logic.

6. The vocabulary of Set Theory includes the single binary relation symbol \(\in\), which is intended to express membership, and the usual identity relation symbol.

Express in first-order logic the following axioms:

- There exists an empty set.
  \[(\exists s)(\forall x)\neg(x \in s)\]

- Two sets are identical iff they have precisely the same members.
  \[(\forall s)(\forall s')(((s \approx s') \leftrightarrow (\forall x)(x \in s \leftrightarrow x \in s')))\]

- For every set \(S\), there is a set \(S'\) that contains \(S\) strictly.
  \[(\forall s)(\exists s')(\forall x)(x \in s \rightarrow x \in s') \land (\exists x)(\neg(x \in s) \land x \in s'))\]