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THE IMPLICATION PROBLEM FOR FUNCTIONAL AND INCLUSION DEPENDENCIES IS UNDECIDABLE*

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Abstract. The implication problem for a class of dependencies is the following: given a finite set of dependencies, determine if they logically imply another dependency. We show that the implication problem is undecidable for the class of functional and inclusion dependencies. This holds true even if the inclusion dependencies are restricted to be binary. It may be noted that the implication problem is known to be decidable for functional and unary inclusion dependencies and also for inclusion dependencies without functional dependencies.

Key words. axiomatization, data base dependency, functional dependency, inclusion dependency, recursive inseparability, relational data base, undecidability

1. Introduction. Functional and inclusion dependencies are the most fundamental database integrity constraints, and they are used in essentially all data models. Their interaction has recently investigated in several papers [CFP], [FV], [JK].

In order to utilize dependencies in the database design process one needs to be able to test for logical implication, i.e., does a set of dependencies logically imply another dependency [Be]. The implication problem is one of the prominent issues in dependency theory.

It is known that, when only functional dependencies are given or when only inclusion dependencies are given, the implication problem is decidable [BB], [CFP]. In this note we show that when functional and inclusion dependencies are considered simultaneously the problem becomes undecidable. (This result was also obtained independently by Mitchel [M2].) We also show that implication for functional and inclusion dependencies is reducible to implication for functional and binary inclusion dependencies, and we study the consequences of the undecidability result on the issue of obtaining an effective axiomatization for implication.

2. Definitions. A *relation scheme* U is a finite sequence $C_1, \cdots, C_n$ of *attributes*, which, intuitively, serve as column headings. A *tuple* t over U is a sequence $\langle c_1, \cdots, c_n \rangle$ of the same length as U. A *relation* R over U is a set (not necessarily finite) of tuples over U. U is called the *scheme* of R. If $C_{i_1}, \cdots, C_{i_m}$ is a sequence of members of U, then

$$t[C_{i_1}, \cdots, C_{i_k}] = \langle c_{i_1}, \cdots, c_{i_k} \rangle,$$

and

$$R[C_{i_1}, \cdots, C_{i_k}] = \{t[C_{i_1}, \cdots, C_{i_k}]: t \in R\}.$$

A *functional dependency* [Co] (abbr. *fd*) is a statement $A_1, \cdots, A_k \to B_1, \cdots, B_l$, where $k, l \geqq 0$ and the A's and B's are attributes. A relation R whose scheme includes $A_1, \cdots, A_k, B_1, \cdots, B_l$ *satisfies* this fd if, for all tuples $s, t \in R$, if $s[A_1, \cdots, A_k] = t[A_1, \cdots, A_k]$ then $s[B_1, \cdots, B_l] = t[B_1, \cdots, B_l]$.

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An *inclusion dependency* [Fa] (abbr. *ind*) is a statement $A_1, \cdots, A_k \subseteq B_1, \cdots, B_k$, where the A's and B's are attributes. A relation R whose scheme includes $A_1, \cdots, A_k$, $B_1, \cdots, B_k$ *satisfies* this ind if $R[A_1, \cdots, A_k] \subseteq R[B_1, \cdots, B_k]$.

A relation R satisfies a set D of dependencies if it satisfies all dependencies in D. A set D is said to *imply* a dependency d, denoted $D \vDash d$, if d is satisfied by all relations that satisfy D. D is said to *finitely imply* d, denoted $D \vDash_f d$, if d is satisfied by all finite relations that satisfy D. Clearly, if $D \vDash d$ then also $D \vDash_f d$. But it is shown in [CFP] that the converse is not always the case. From a practical point of view, finite implication is the more interesting notion.

The *implication problem* for fd's and ind's is to decide, given a finite set D of fd's and ind's and an fd or an ind d, whether $D \vDash d$. The *finite implication problem* is to decide whether $D \vDash_f d$. In the following section we show that both problems are undecidable.

3. The main result. The problem that we use for our undecidability proof is the word problem for (finite) monoids. Recall that a monoid is an algebra with an associative binary operation $\cdot$ and a unit element **1**. Let Σ be an alphabet. Σ^* is the free monoid generated by Σ. Now let $E = \{\alpha_i = \beta_i : 1 \leqq i \leqq n\}$ be a finite set of equalities, and let e be an additional equality $\alpha = \beta$, where $\alpha, \beta, \alpha_i, \beta_i \in \Sigma^*$. We say that E (*finitely*) *implies* e, denoted $E \vDash e$ ($E \vDash_f e$), if for every (finite) monoid M and homomorphism $h: \Sigma^* \to M$, if $h(\alpha_i) = h(\beta_i)$ for $1 \leqq i \leqq n$, then also $h(\alpha) = h(\beta)$. The *word problem* for (finite) monoids is to decide, given E and e, whether $E \vDash e$ ($E \vDash_f e$). The word problem for monoids was shown to be undecidable in [Po] and the word problem for finite monoids was shown to be undecidable in [Gu]. These results also follow from a recursive inseparability result of [GL]. Two sets Φ and Ψ are *recursively inseparable* if there is no recursive set Π such that $\Phi \subseteq \Pi$ but Ψ and Π are disjoint. If Φ and Ψ are recursively inseparable then clearly they are both not recursive.

THEOREM 1. [GL] *The set* $\{(E, e) : E \vDash e\}$ *is recursively inseparable from the set* $\{(E, e) : E \nvDash_f e\}$, *where E ranges over finite sets of equalities and e ranges over equalities.*

THEOREM 2. *The set* $\{(D, d) : D \vDash d\}$ *is recursively inseparable from the set* $\{(D, d) : D \nvDash_f d\}$, *where D ranges over finite sets of* fd's *and* ind's *and d ranges over* ind's.

Proof. We reduce the word problem for (finite) monoids to (finite) implication of ind's.

Let Σ be our alphabet. Let $E = \{\alpha_i = \beta_i : 1 \leqq i \leqq n\}$ be a set of equalities, and let e be another equality $\alpha = \beta$. We call every prefix of α, β, α_i, or β_i, for $1 \leqq i \leqq n$, a *prefix* (note that the null string λ is a prefix, and so is α, etc.). We use a relation R whose relation scheme has the following attributes:

(1) γ, for each prefix γ;
(2) X, Y, and Z;
(3) Ya, for each $a \in \Sigma$;
(4) Za, for each $a \in \Sigma$.

Intuitively, the γ's represent the corresponding elements of the monoid, X and Y represent arbitrary elements with Z as their product, and the Ya and Za represent multiplying by a from the right.

The set D consists of the following dependencies:

(1) $\lambda, X, Y \to Z$, and $\lambda, Y \to Ya$, for each $a \in \Sigma$.
(2) $\lambda, \lambda \subseteq \lambda, Y$.
(3) $\lambda, \gamma, \gamma a \subseteq \lambda, Y, Ya$, for each $a \in \Sigma$ and pair of prefixes γ, γa.
(4) $\lambda, Z, Za \subseteq \lambda, Y, Ya$, for each $a \in \Sigma$.
(5) $\lambda, X, Ya, Za \subseteq \lambda, X, Y, Z$, for each $a \in \Sigma$.

(6) $\lambda, Y, \lambda, Y \subseteq \lambda, X, Y, Z$.
(7) $\lambda, \alpha_i \subseteq \lambda, \beta_i$.

The above dependencies ensure that the attributes behave according to the intended meaning. Note that D does not guarantee uniqueness of the elements that correspond to the prefixes. We could have added the fd $\varnothing \to \lambda$, but we did not want to use fd's with empty left-hand side, and we are willing to pay for it with some complication in the reduction.

Finally, the dependency d is $\alpha \subseteq \beta$.

CLAIM. $D \models d$ (*resp.* $D \models_f d$) iff $E \models e$ (*resp.* $E \models_f e$).

Proof of claim.

Only-if part. We show that $E \not\models e$ (resp. $E \not\models_f e$) entails $D \not\models d$ (resp. $D \not\models_f d$). Suppose that $E \not\models e$. Then there is a homomorphism h on Σ^* such that $h(\alpha_i) = h(\beta_i)$ but $h(\alpha) \neq h(\beta)$. For a pair x, y of elements in Σ^* we define a tuple $t_{x,y}$ by:

$t_{x,y}[\gamma] = h(\gamma)$, for each prefix γ,
$t_{x,y}[X] = h(x)$,
$t_{x,y}[Y] = h(y)$,
$t_{x,y}[Ya] = h(ya)$, for each $a \in \Sigma$,
$t_{x,y}[Z] = h(xy)$, and
$t_{x,y}[Za] = h(xya)$, for each $a \in \Sigma$.

Let now $R = \{t_{x,y}: x, y \in \Sigma^*\}$. We leave it to the reader to show that R satisfies D but not d. If $E \not\models_f e$, then in addition we can assume that $\{h(x): x \in \Sigma^*\}$ is finite, so R is also finite.

If part. Assume that $E \models e$, and let R be a relation satisfying D. Let $s \in R$. We have to show that $s[\alpha] \in R[\beta]$. We now define a homomorphism h of Σ^* into the domain of the relation R, such that $h(\lambda) = s[\lambda]$. For an element $y \in \Sigma^*$, we define the element $h(y)$ and show that $\langle h(\lambda), h(y)\rangle \in R[\lambda, Y]$ inductively as follows:

(1) $h(\lambda)$ is $s[\lambda]$. Note that $\langle h(\lambda), h(\lambda)\rangle \in R[\lambda, Y]$ by the ind $\lambda, \lambda \subseteq \lambda, Y$.

(2) Suppose that we have defined $h(y)$ and showed that $\langle h(\lambda), h(y)\rangle \in R[\lambda, Y]$, and let $a \in \Sigma$. Define $h(ya)$ to be the unique element $t[Ya]$ for the tuple t such that $t[\lambda, Y] = \langle h(\lambda), h(y)\rangle$ (uniqueness is ensured by the fd $\lambda, Y \to Ya$). $\langle h(\lambda), h(ya)\rangle \in R[\lambda, Y]$ by the ind $\lambda, X, Ya, Za \subseteq \lambda, X, Y, Z$.

Next we show that for all $x, y \in \Sigma^*$, we have that $\langle h(\lambda), h(x), h(y), h(xy)\rangle \in R[\lambda, X, Y, Z]$. This is shown by induction on y. We have already shown that $\langle h(\lambda), h(x)\rangle \in R[\lambda, Y]$, so by the ind $\lambda, Y, \lambda, Y \subseteq \lambda, X, Y, Z$ we have $\langle h(\lambda), h(x), h(\lambda), h(x)\rangle \in R[\lambda, X, Y, Z]$. Assume now that $\langle h(\lambda), h(x), h(y), h(xy)\rangle \in R[\lambda, X, Y, Z]$. Let $t \in R$ be such that $t[\lambda, X, Y, Z] = \langle h(\lambda), h(x), h(y), h(xy)\rangle$. Then by definition $t[Ya] = h(ya)$. Also by the ind $\lambda, Z, Za \subseteq \lambda, Y, Ya$ we have that $t[Za] = h(xya)$. By the ind $\lambda, X, Ya, Za \subseteq \lambda, X, Y, Z$ we have that

$$\langle h(\lambda), h(x), h(ya), h(xya)\rangle \in R[\lambda, X, Y, Z].$$

We now define a binary operation $\cdot$ on the set $\{h(x): x \in \Sigma^*\}$. Let $x, y \in \Sigma^*$. The above argument shows that there is a tuple t in R such that $t[\lambda, X, Y] = \langle h(\lambda), h(x), h(y)\rangle$. We define $h(x) \cdot h(y)$ as $t[Z]$, which is unique by the fd $\lambda, X, Y \to Z$. Furthermore, we have shown that $t[Z]$ is just $h(xy)$, so it follows that h is a homomorphism on Σ^*.

Now, using the fd's $\lambda, Y \to Ya$ and the ind's $\lambda, \gamma, \gamma a \subseteq \lambda, Y, Ya$, it is easy to show by induction that R satisfies $\lambda \to \gamma$ and that $s[\gamma] = h(\gamma)$, for every prefix γ. In particular, $s[\alpha_i] = h(\alpha_i)$ and $s[\beta_i] = h(\beta_i)$. By the ind $\lambda, \alpha_i \subseteq \lambda, \beta_i$, there is a tuple $t \in R$ such that $t[\lambda, \beta_i] = \langle h(\lambda), h(\alpha_i)\rangle$. Since R satisfies $\lambda \to \beta_i$ it must be the case that $h(\alpha_i) = h(\beta_i)$.

Since $E \models e$, we have that $h(\alpha) = h(\beta)$, and since $s[\alpha] = h(\alpha)$ and $s[\beta] = h(\beta)$, it follows that $s[\alpha] \in R[\beta]$. For the finite case, if R is a finite relation, then $\{h(x): x \in \Sigma^*\}$ is finite, and the rest of the argument is the same. □

COROLLARY. *The sets* $\{(D, d): D \models d\}$ *and* $\{(D, d): D \not\models_f d\}$, *where* D *ranges over finite sets of* fd's *and* ind's *and* d *ranges over* ind's *are not recursive.*

The proof above showed that testing whether a set of fd's and ind's implies or finitely implies an ind is undecidable. How about testing (finite) implication of fd's? The next theorem shows that this is also undecidable.

THEOREM 3. *The set* $\{(D, d): D \models d\}$ *is recursively inseparable from the set* $\{(D, d): D \not\models_f d\}$, *where* D *ranges over finite sets of* fd's *and* ind's *and* d *ranges over* fd's.

Proof. We reduce the word problem for (finite) monoids to (finite) implication of fd's. Given Σ, E, and e we construct the set D of fd's and ind's as in the proof of Theorem 2 with the addition of the attributes α', β', and Y', and the dependencies $Y \to Y'$, $\alpha, \alpha' \subseteq Y, Y'$, and $\beta, \beta' \subseteq Y, Y'$. The dependency d is $\alpha' \to \beta'$.

CLAIM. $D \models d$ (*resp.* $D \models_f d$) *iff* $E \models e$ (*resp.* $E \models_f e$).

Proof of Claim.

If part. Assume $E \models e$, and let R be a relation satisfying D. Let $s, t \in R$ such that $s[\alpha'] = t[\alpha']$. We have to show that $s[\beta'] = t[\beta']$. The If Part in the proof of Theorem 2 shows that $s[\beta] = s[\alpha]$ and $t[\alpha] = t[\beta]$. Because of the ind's $\alpha, \alpha' \subseteq Y, Y'$ and $\beta, \beta' \subseteq Y, Y'$, there are tuples $s_1, s_2 \in R$ such that $s[\alpha, \alpha'] = s_1[Y, Y']$, and $s[\beta, \beta'] = s_2[Y, Y']$. It follows that $s[\alpha'] = s[\beta']$, because of the fd $Y \to Y'$. Similarly, $t[\alpha'] = t[\beta']$. Therefore, $s[\beta'] = t[\beta']$.

Only-if part. Suppose that $E \not\models e$. Then there are a monoid M and a homomorphism $h: \Sigma^* \to M$ such that $h(\alpha_i) = h(\beta_i)$ but $h(\alpha) \neq h(\beta)$. For each element $y \in M$, let y' be a new distinct element. We construct R for h as in the proof of Theorem 2 with the additional clauses:

$t_{x,y}[\alpha'] = (h(\alpha))'$,

$t_{x,y}[\beta'] = (h(\beta))'$, and

$t_{x,y}[Y'] = (h(y))'$.

Clearly, R satisfies D.

Let $\tilde{M}$ be an isomorphic disjoint copy of M, and $\tilde{h}$ the corresponding homomorphism from Σ^*. Again, for each element $y \in \tilde{M}$, let y' be a new distinct element, such that for all $x \in M$ and $y \in \tilde{M}$, we have that $x' = y'$ iff $x = h(\alpha)$ and $y = \tilde{h}(\alpha)$. Let $\tilde{R}$ be constructed from $\tilde{h}$ in a manner analogous to the construction of R from h. We leave it to the reader to verify that $R \cup \tilde{R}$ satisfies D but not d. Also, if $E \not\models_f e$, then we can assume that $R \cup \tilde{R}$ is finite. □

COROLLARY. *The sets* $\{(D, d): D \models d\}$ *and* $\{(D, d): D \not\models_f d\}$, *where* D *ranges over finite sets of* fd's *and* ind's *and* d *ranges over* fd's, *are not recursive.*

4. Reduction to binary inclusion dependencies. An ind $A_1, \cdots, A_k \subseteq B_1, \cdots, B_k$ is said to be *m-ary* if $k \leqq m$. The reduction in the previous section uses 4-ary ind's. It follows that the (finite) implication problem is undecidable even when restricted to 4-ary ind's. On the other hand, in [KCV] it is shown that both the implication and the finite implication problems are decidable when restricted to unary ind's. In this section we close the gap between those two results by showing that (finite) implication of fd's and ind's is reducible to (finite) implication of fd's and binary ind's.

THEOREM 4. *There is an algorithm that, given a set* D *of* fd's *and* ind's *and an* fd (*resp.* ind) d, *produces a finite set* D' *of* fd's *and binary* ind's *and an* fd (*resp. a unary ind*) d', *such that* $D \models d$ *if and only if* $D' \models d'$ *and* $D \models_f d$ *if and only if* $D' \models_f d'$.

Proof. Let all the ind's in $D \cup \{d\}$ be m-ary. Furthermore, we can assume without loss of generality that the left-hand side and right-hand side of these ind's contain exactly m attributes (we can always duplicate attributes to achieve that). We denote a sequence $A_1, \cdots, A_m$ of attributes by $\bar{A}$. We view a sequence as a list of elements.

When we enclose the sequence in parentheses, e.g., $(\bar{A})$, we refer to it as an element in the domain of sequences.

We first construct a set F of fd's and ind's. We introduce new attributes $X_1, \cdots, X_m, X$, and put in F the fd's $\bar{X} \to X$ and $X \to \bar{X}$. For every ind $\bar{A} \subseteq \bar{B}$ in $D \cup d$, we introduce new attributes $(\bar{A})$ and $(\bar{B})$ and put in F the binary ind's $A_i, (\bar{A}) \subseteq X_i, X$ and $B_i, (\bar{B}) \subseteq \bar{X}, X$, for $1 \leqq i \leqq m$. Let τ be the ind $\bar{A} \subseteq \bar{B}$. We define τ' as the unary ind $(\bar{A}) \subseteq (\bar{B})$. If τ is an fd then we let $\tau' = \tau$. Now we define $D' = F \cup \{\tau' : \tau \in D\}$.

CLAIM 1. *For all* $\tau \in D \cup \{d\}$ *we have* $F \cup \{r'\} \models \tau$.

Proof of claim. The claim is trivially true for fd's. Let τ be the ind $\bar{A} \subseteq \bar{B}$. Let R be a relation that satisfies $F \cup \tau'$, and let $t \in R$. We have to show that there is a tuple $s \in R$ such that $s[\bar{B}] = t[\bar{A}]$. Since R satisfies τ', there is a tuple $s \in R$ such that $s[(\bar{B})] = t[(\bar{A})]$. Since R satisfies F there are tuples $t_i, s_i \in R$ such that $t_i[X_i, X] = t[A_i, (\bar{A})]$ and $s_i[X_i, X] = s[B_i, (\bar{B})]$. But, since $s[(\bar{B})] = t[(\bar{A})]$, and R satisfies F, it follows that $t[A_i] = s[B_i]$. This is true for $1 \leqq i \leqq m$, which proves the claim.

CLAIM 2. *For all* $\tau \in D \cup d$ *we have* $F \cup \{\tau\} \models \tau'$.

Proof of claim. The claim is trivially true for fd's. Let τ be the ind $\bar{A} \subseteq \bar{B}$. Let R be a relation that satisfies $F \cup \{\tau\}$, and let $t \in R$. We have to show that there is a tuple $s \in R$ such that $s[(\bar{B})] = t[(\bar{A})]$. Since R satisfies F there are tuples $t_1, \cdots, t_m \in R$ such that $t[A_i, (\bar{A})] = t_i[X_i, X]$. Furthermore, $t_1[X_i] = t[A_i]$, for $1 \leqq i \leqq m$, so $t_1[\bar{X}, X] = t[\bar{A}, (\bar{A})]$. Since R satisfies τ, there is a tuple $s \in R$ such that $s[\bar{B}] = t[\bar{A}]$. Again, since R satisfies F, we can show that there is a tuple $s_1 \in R$ such that $s_1[\bar{X}, X] = s[\bar{B}, (\bar{B})]$. It follows that $t_1[X] = s_1[X]$, so $t[(\bar{A})] = s[(\bar{B})]$. This proves the claim.

We can now prove the theorem.

Only-if part. Assume $D \models d$, and let R be a relation satisfying D'. By Claim 1, we have that R satisfies D, so by assumption R satisfies d. Since R satisfies F and d, it satisfies d' by Claim 2.

If part. Assume $D \not\models d$, and let R be a relation satisfying D but not d. We construct a relation R' satisfying D' but not d'. Let U be the sequence of attributes that are used in dependencies in $D \cup \{d\}$. Let U^m be a sequence consisting of all sequences of length m of attributes from U. We use "," to denote concatenation of sequences. R_1 is a relation on U, U^m defined as:

$$\{t : t[U] \in R, t[(\bar{A})] = (t[\bar{A}]) \text{ for } \bar{A} \in U^m\}.$$

R_2 is a relation on $\bar{X}, X$ defined as

$$\{t : t \in R_1[\bar{A}, (\bar{A})] \text{ for some } \bar{A} \in U^m\}.$$

Finally, R' is a relation on $U, U^m, \bar{X}, X$ defined as

$$\{t : t[U, U^m] \in R_1 \text{ and } t[\bar{X}, X] \in R_2\}.$$

We leave it to the reader to show that R' satisfies F. Since $R'[U] = R[U]$, it follows that R' also satisfies D. By Claim 2 it follows that R' satisfies D'. Suppose now that R' satisfies d'. Then Claim 1 entails that R' satisfies d, and consequently that R satisfies d, contrary to the assumption. Thus R' does not satisfy d'.

Both parts of the proof work also for the finite case once we observe that the construction of R' from R preserves finiteness. □

COROLLARY. *The implication and the finite implication problems for* fd'*s and binary* ind'*s are undecidable.*

5. Axiomatization. Parallel to the pursuit for algorithms to test implication is the pursuit for axiomatization of implication. That is, while it might be impossible to recursively check whether $D \models d$, it might be possible to recursively check whether a given proof that $D \models d$ is correct. An immediate consequence of the existence of an axiomatization for a problem is the partial decidability of the problem; one has just to generate methodically all possible proofs and check for their correctness.

Consider now finite implication for fd's and ind's. It is not hard to see that the set $\{(D, d): D \not\models_f d\}$ is recursively enumerable; just enumerate all finite relations and check whether they satisfy D but not d. We have shown, however, that this set is not recursive. It follows that the set $\{(D, d): D \models_f d\}$ is not even recursively enumerable. Consequently, there can be no axiomatization for finite implication of fd's and ind's (an axiomatization for finite implication of fd's and unary ind's is described in [KCV]).

The situation with implication for fd's and ind's is different. Both fd's and ind's can be expressed by first-order sentences, and implication is reducible to validity of a first-order sentence. It follows that the set $\{(D, d): D \models d\}$ is recursively enumerable, and this raises the question whether implication of fd's and ind's can be axiomatized.

Casanova et al. [CFP] gave a partial negative answer to this question. They have shown that there is no k-ary axiomatization for implication of fd's and ind's, i.e., there is no axiomatization where the number of premises in the inference is bounded by some number k. (In contrast, fd's alone and ind's alone are known to have binary axiomatizations [Ar], [CFP].) In proving this result Casanova et al. assumed that axioms and inference rules do not introduce new attributes. That is, if π is a proof that $D \models d$, then only attributes that occur in dependencies in D or d occur in π. We call such an axiomatization *attribute-bounded* (it is called *universe-bounded* in [Va]). The axiomatizations for implication of fd's in [Ar] and for implication of ind's in [CFP] are both attribute-bounded. Mitchell [M1], on the other hand, has shown that by dropping the attribute-boundedness requirement we can get a ternary axiomatization for implication of fd's and ind's.

The question remained open whether there is an attribute-bounded (but non-k-ary) axiomatization for implication of fd's and ind's. (For an example of a non-k-ary axiomatization see [BV], [KCV].) Our undecidability results entail that the answer to this question is negative. Suppose to the contrary that there is such an axiomatization. Given a finite set of fd's and ind's D, and an fd or an ind d, there are only finitely many possible proofs for the implication $D \models d$ that uses only attributes that occur in D or d. By checking all these proofs, we can determine whether indeed $D \models d$-contradiction.

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