Sets and Subsets

Countable and Uncountable
Reading

Appendix A

- Section A.6.8
- Pages 788-792
BIG IDEAS

Themes

1. There exist functions that cannot be computed in Java or any other computer language.

2. There exist subsets of the Natural Numbers that cannot be described in English or any other human language.

3. There exist languages that cannot be recognized by any finite state machine.
Basic Concepts

1. Set
2. Subset
3. Function
Notation

Functions

• \( B^A = \{ f \mid f : A \rightarrow B \} \)

Example

• \( 2 = \{ 0, 1 \} \)

• \( 2^A = \{ f \mid f : A \rightarrow \{ 0, 1 \} \} \)
Functions, Bitstrings, and Subsets

Main Observations

• Functions, Bitstrings, and Subsets are the same things

Functions, Bitstrings, and Subsets

• \(2^A = \{ f \mid f : A \to \{0,1\} \} = \{S \mid S \subseteq A\}\) -- Functions and Bitstrings

-- \(\{f \leftrightarrow S\} \iff \{a \in S \iff f(a) = 1\}\) -- Subsets

Example: Bitstring Representation of Subsets

• \(A = \{a_1, \ldots, a_{10}\}\)

• \(S = \{a_2, a_4, a_5, a_9\}\)

• \(b_S = 0101100010\)
Cardinality

• \(|A| = \text{cardinality of } A = \text{the number of elements in the set } A\)

Power Set

• \(P(A) = \{B \mid B \subseteq A\} = \text{set of all subsets of } A\)
• \(2^A = \{f \mid f : A \rightarrow \{0,1\}\}\)
• \(P(A) \equiv 2^A\)
  -- \(|P(A)| \equiv 2^{|A|}\) -- proof by binomial enumeration
  -- \(|2^A| = 2^{|A|}\)
Theorem: Any set $A$ with $n$ elements has exactly $2^n$ subsets including itself and the empty set. Thus $|P(A)| \equiv 2^{|A|}$

Proof: There are $2^n$ binary strings of length $n$.

Each binary string represents a unique subset.

Therefore $|P(A)| \equiv 2^{|A|}$.
Russell’s Paradox

1. Type 1 Sets = Sets that contain themselves as Elements
   - Example: Set consisting of all sets with 3 or more Elements

2. Type 2 Sets = Sets that do not contain themselves as Elements
   - N, Z, ...

3. Let $S = \text{All Sets of Type 2} = \text{Set of all sets not containing themselves as Elements}$
   - $S \in S \rightarrow S \text{ is Type 1} \rightarrow S \notin S$       CONTRADICTION
   - $S \notin S \rightarrow S \text{ is Type 2} \rightarrow S \in S$       CONTRADICTION

   But every element must either be in $S$ or not in $S$!

   - Axiomatic Set Theory introduced to control these paradoxes.
Countable and Uncountable Sets
There exist non-computable functions.

There exist subsets of the Natural Numbers that we cannot describe.

There exist languages that cannot be recognized by any finite state machine.
Countable Sets

Finite

- List Comes to an End

Countably Infinite

- List Does NOT Come to an End
  -- No Last Number
  -- Infinite List
  -- 1-1 correspondence with $N$
Infinity

Infinity (∞) is NOT a natural number

-- ∞ + 1 = ∞

-- ∞ + ∞ = ∞

-- ∞ × ∞ = ∞

-- ∞^n = ∞
Cardinality

Size

• There are many different notions of relative size:
  -- Subset
  -- Length
  -- 1-1 Correspondence

• These notions are NOT the same.

• Cardinality deals with 1-1 correspondence.

• Size is measured by bijection NOT by subset!
Countable Sets

1. $N$ -- Natural Numbers

2. $N \cup \{-1\}$

3. Even numbers, Odd numbers

4. $Z$ -- Integers (Positive and Negative)

5. $N \times N$ -- Pairs of Integers

6. $Q$ -- Rational Numbers

7. $A \times B$ -- if $A,B$ are both countable

8. $A_1 \times \cdots \times A_n$ -- if $A_1, \ldots, A_n$ are all countable

9. $B \supset A$, and $B$ countable $\implies$ $A$ countable
Tricks for Proving Countability

1. Shifts
   - $-1, 0, 1, 2, \ldots$
   - Even numbers

2. Interleave -- $\mathbb{Z}$

3. Doubly Infinite Patterns -- $N \times N$

4. Bijection from $N$ or $A \times B$, where $A, B$ are countable

5. Subset of a countable set

6. Intuition: Countable means there is a pattern
Theorem 0: \( N \times N \) is countable.

Proof: List all the pairs in infinite horizontal rows:

\[
\begin{align*}
(0,0) & \quad (0,1) \quad (0,2) \ldots \\
(1,0) & \quad (1,1) \quad (1,2) \ldots \\
(2,0) & \quad (2,1) \quad (2,2) \ldots \\
\vdots & \\
\end{align*}
\]

List along the diagonals.

Every pair will eventually appear in the list.
Rational Numbers

Theorem 0: $N \times N$ is countable.

Corollary 0: The rational numbers $\mathbb{Q}$ are countable.

Proof: Every rational number can be represented by a pair of natural numbers.
Theorem 1: If \( \Sigma \) is a finite alphabet, then \( \Sigma^* \) is countable.

Proof: Order the elements of \( \Sigma^* \) by length of the string, lexicographically.

\[
\Sigma^* = \{ \varepsilon, a, \ldots, z, aa, \ldots, zz, \ldots \}
\]

Clearly we list every element in \( \Sigma^* \) in a finite number of steps.

Examples

- Set of All English sentences
- Set of All JAVA programs
- All the sets of interest in Computer Science are countable.
Theorem 2: \( P(N) \) is uncountable.

Proof: Diagonalization argument.

Theorems 3: \( \mathbb{R} \) is uncountable.

Proof: Diagonalization argument on \([0,1]\).

Diagonalization argument based on power set representation.

Remark: Notice that \( \mathbb{R} \neq \Sigma^* \) where \( \Sigma = \{0,1,\ldots,9\} \).

Corollary: There exist irrational numbers.
**Theorem 2:** \( P(N) \) is uncountable.

**Proof:** By Contradiction. Suppose that \( P(N) \) is countable.

- Let \( S_1, S_2, \ldots \) be a list of ALL the subsets of \( N \).
- Define: \( S = \{ n \in N \mid n \notin S_n \} \)
- If \( S = S_n \), then there are 2 cases:

  **Case 1**
  
  \[ n \notin S_n \Rightarrow n \notin S \quad \text{(since } S = S_n) \]
  
  \[ n \notin S_n \Rightarrow n \in S \quad \text{(by definition of } S) \]

  Impossible.

  **Case 2**

  \[ n \in S_n \Rightarrow n \in S \]
  
  \[ n \in S_n \Rightarrow n \notin S \]

  Impossible.

Therefore \( S \subseteq N \) is not in the list, so there is no list containing all the subsets of \( N \).
Theorems 3: \( \mathbb{R} \) is uncountable.

Proof 1: By Contradiction. Suppose that \([0,1] \subset \mathbb{R}\) is countable.

\[
1 \leftrightarrow .d_{11}d_{12} \ldots \\
2 \leftrightarrow .d_{21}d_{22} \ldots \\
\vdots \\
n \leftrightarrow .d_{n1}d_{n2} \ldots d_{nn} \ldots \\
\vdots
\]

Define: \( b = b_1b_2 \ldots b_n \ldots \), where \( b_n \neq d_{nn} \) for ALL \( n \).

Then \( b \) is not in the list!

Hence there is no list containing all the numbers in \( \mathbb{R} \).
Theorems 3: $\mathbb{R}$ is uncountable.

Proof 2: By Construction: $[0,1] = 2^\mathbb{N} = P(\mathbb{N})$, since every real number in $[0,1]$ has a representation in binary.

$$b = .b_1b_2\cdots b_n\cdots \leftrightarrow \text{subset of } \mathbb{N}$$

But $P(\mathbb{N})$ is uncountable, so $\mathbb{R}$ is uncountable.

Corollary: $\mathbb{R} = 2^\mathbb{N} = P(\mathbb{N})$
Uncountable Sets

Examples

• \( \mathbb{R} \)
• \( P(\mathbb{N}) \)
• \( B \supseteq A \) and \( A \) uncountable \( \implies \) \( B \) uncountable

Tricks for Proving Uncountability

• Diagonalize
• Superset of an uncountable set
• Bijection from an uncountable set

Intuition

• Uncountable means there is no pattern
**Non-Computable Functions**

*Theorem 4:* The functions $f : \mathbb{N} \to \{0,1\}$ are uncountable.

*Proof:* $2^\mathbb{N} = P(\mathbb{N})$.

*Corollary 1:* There exist non-computable functions.

*Corollary 2:* There exist subsets of the Natural Numbers that cannot be described in any language.
Non-Computable Functions -- Revisited

Theorem 4: The functions $f : \mathbb{N} \to \{0,1\}$ are uncountable.

Theorem 4a: The subsets $S \subseteq \mathbb{N}$ are uncountable.

Corollary 1*: Almost all functions on $N$ are non-computable.

Corollary 2*: Almost all subsets of the Natural Numbers cannot be described.
Theorem 5: The countable union of countable sets is countable.

Proof: Consider the union of $A_1, A_2, \ldots, A_k, \ldots$.

Since each set is countable, we can list their elements.

- $A_k = a_{k,1}, a_{k,2}, \ldots$

Now proceed as in the proof that $N \times N$ is countable:

\[
\begin{array}{cccc}
   a_{(1,1)} & a_{(1,2)} & a_{(1,3)} & \cdots \\
   a_{(2,1)} & a_{(2,2)} & a_{(2,3)} & \cdots \\
   a_{(3,1)} & a_{(3,1)} & a_{(3,3)} & \cdots \\
   & \vdots & & \\
\end{array}
\]

Eventually every element in the union appears in the list.

Equivalently map $f : N \times N \to \bigcup A_n$ Union by setting $f(i, j) = a_{i,j}$. 


Consequences

Corollaries

• The set of algebraic numbers -- solutions of polynomial equations -- is countable because the polynomials are countable and every polynomial has finitely many solutions.

• There exist transcendental numbers -- numbers that are not the solutions of polynomial equations -- because the real numbers are not countable.

Observation

• The countable product of countable sets is NOT countable because $\mathbb{R}$ is not countable.
Languages and Finite State Automata

Languages

• \( \Sigma = \) finite alphabet

• \( \Sigma^* = \) all finite strings of \( \Sigma \) is countable

• \( L \subseteq P(\Sigma^*) \) -- Language

• \# Languages = \# Subsets of \( \Sigma^* \) -- Uncountable

Finite State Automata

• \( Q = \) finite set of states

• \( \Sigma = \) finite set of input symbols

• \( M = \{Q, \Sigma, \delta, q_0, F\} = \) finite state automaton

• \# Finite State Automata -- Countable
Cardinality of Finite Automata

Finite Automata

- $Q$ = finite set of states
- $\Sigma$ = finite set of input symbols
  -- $|Q|$ possible starting states
  -- $2^{|Q|}$ possible accepting states
  -- $|Q|^{|\Sigma|+1}$ possible transition functions

- Total number of finite automata with $|Q|$ states and $|\Sigma|$ input symbols
  -- $|Q| \times 2^{|Q|} \times |Q|^{|\Sigma|+1} = 2^{|Q|} \times |Q|^{|\Sigma|+2}$

- Total number of finite automata is countable
  -- Countable union of countable sets
Another Paradox

Theorem 5

• No set is 1-1 with its power set.

Paradox

• $S = \text{Set of all sets}$
• $P(S)$ is larger than $S$
• But $S$ is everything!
Continuum Hypothesis

Observations

• $\mathcal{N} = \aleph_0$

• $\mathcal{R} = 2^{\aleph_0}$

Continuum Hypothesis

• $2^{\aleph_0} = \aleph_1$

• Undecidable