Sampling X

$F(x) = Pr(X \le x)$ (CDF)

$\boxed{F(y) \ge F(x) \iff y \ge x}$

$x_1, x_2, \dots, x_n \sim F$

$U[0,1]$

Inversion Method

$F^{-1}(x) = y$

iff $F(y) = x$

Algo:

Draw $Y \sim U[0,1]$

$X = F^{-1}(Y)$

$X \sim F(x)$

To show

$X \sim F(x)$

$Pr(X \le x) = F(x)$

$Pr(F^{-1}(Y) \le x) = F(x)$

$Pr(Y \le F(x)) = F(x)$

Goal: Draw samples from exponential distribution.

$\rightarrow$ $F(x) = 1 - \exp(-\lambda x)$

Draw $Y \sim U[0,1]$

return $-\frac{\log(1-Y)}{\lambda} = \frac{-\log Y}{\lambda}$

Monte Carlo estimation

Goal: $\int_0^{100} e^{-x} \, dx = I$

$= \int_0^1 e^{-x^3} \cdot 1 \, dx = E\left[e^{-U^3}\right]$

$\int_0^1 w(x) \, f(x) \, dx$

Draw $U_1, U_2 \ldots U_N$ from $\boxed{U[0,1]}$

report $\frac{1}{N} \sum_i e^{-U_i^3} = \hat{I}$

$\hat{I} \approx I$

$E(\hat{I}) = I$

Importance Sampling

$$I = \int_0^1 w(x)\,dx.$$

$$\int w(x) \cdot \underset{\substack{\downarrow \\ \text{pdf for uniform distribution}}}{1}\, dx.$$

$$\int_0^1 \frac{w(x)}{g(x)}\, g(x)\, dx$$

$$= E\left[\frac{w(x)}{g(x)}\right]_{g(x)}$$

Algo: Draw

$x_1, x_2 \ldots x_N$ from $g(x)$

report $\frac{1}{N}\sum_i \frac{w(x_i)}{g(x_i)} = \bar{I}$

$E(\bar{I}) = I$

Draw $x_1, x_2 \ldots x_n \sim f(x)$

$$E\left[\frac{w(x)}{}\right.$$

$$\int w(x)\, f(x)\, dx = E[w(x)]_{f(x)}$$

g.

$$= \int w(\cancel{x}) \times \frac{f(x)}{g(x)} \times g(x)\, dx$$

Draw $x_1, x_2 \ldots x_n \sim f(x)$

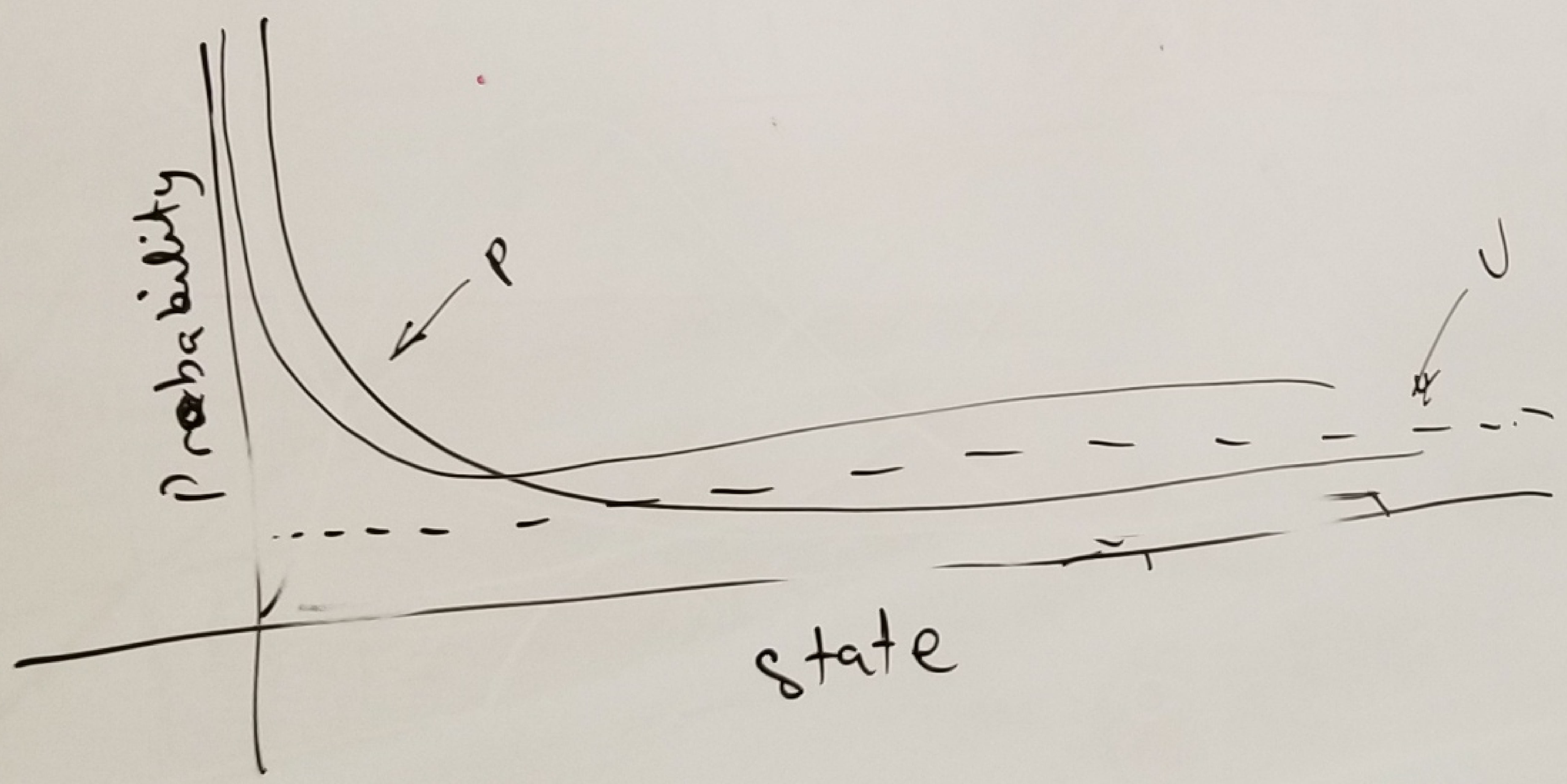
report $\frac{1}{N}\sum_i w(x_i)$

Importance Sampling

Draw $x_1, x_2 \ldots x_n \sim g(y)$

report $\frac{1}{N}\sum_i w(x_i) \times \boxed{\frac{f(x_i)}{g(x_i)}}$

Importance Weight



$$I = \int w(x)\, f(x)\, dx$$

$$\hat{I} = \int w(x) \frac{f(x)}{g(x)} g(x)\, dx$$

$$\mathrm{Var}(\mathrm{Est}) = \frac{1}{N}\left[\int \left(\frac{w^2(x)\, f^2(x)}{g(x)} \right) dx - I^2 \right]$$

$$\int \frac{w^2(x)\, f^2(x)}{g(x)}\, dx \int g(x)\, dx$$

$$\downarrow$$

$$1$$

$$\geq I^2$$

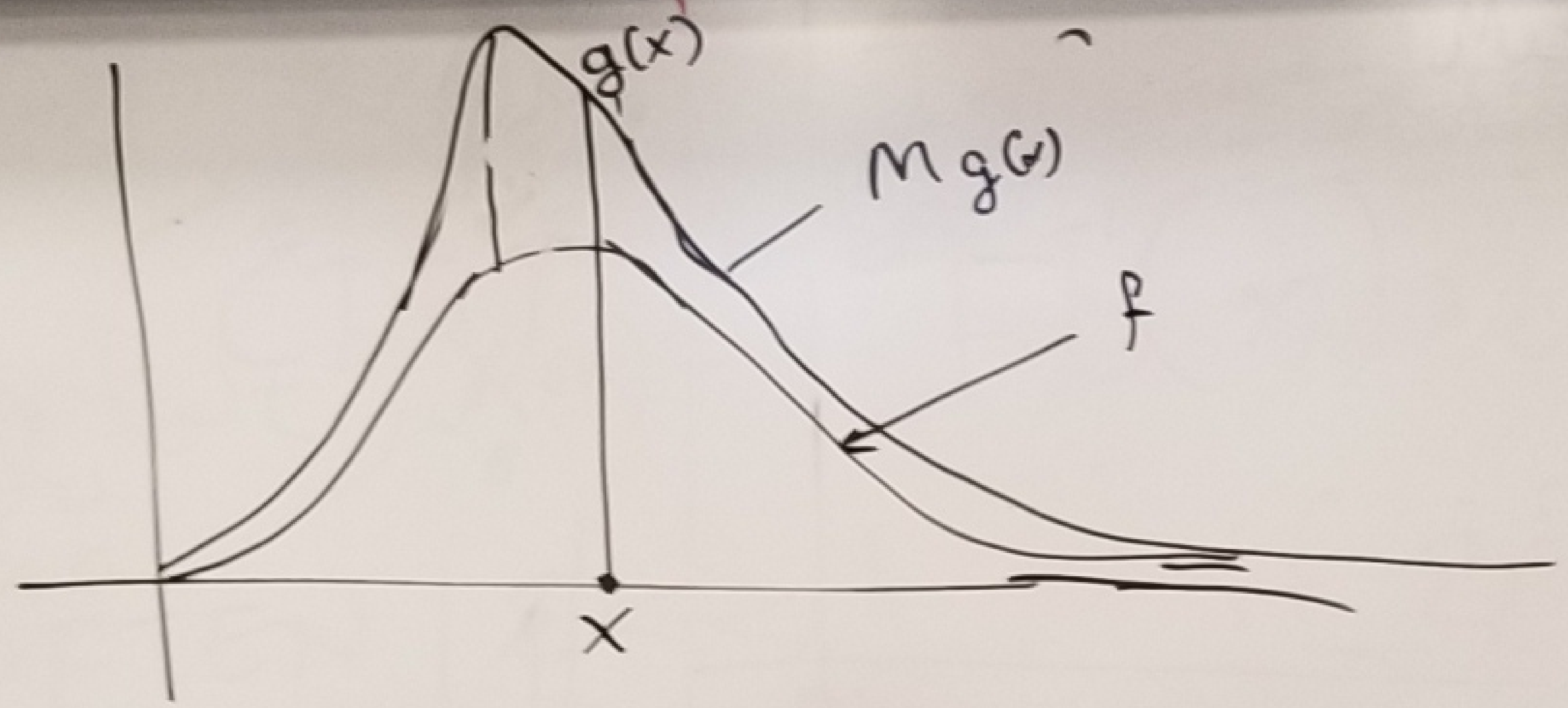
Rejection Sampling.

Goal: Sample from some distribution $f(x)$ [pdf]

Given: we can sample from some $g(x)$ s.t.

$$\boxed{f(x) \leq M \cdot g(x)}$$

$\forall x$



Algorithm.

- Draw $Y \sim g$
- Draw $U \sim U[0,1]$

Accept X if $U \leq \frac{f(y)}{M \cdot g(y)}$

else loop