Rejection Sampling $g(x)$

Goal: Sample from $f(x)$

Given: Ability to sample from $g(x)$ s.t. $f(x) \leq M \cdot g(x) \forall x$ $(M \geq 1)$

Algorithm:
1. Sample $X \sim g(x)$
2. Sample $U \sim U[0, 1]$  
3. Accept $X$ if $U \leq \frac{f(x)}{M \cdot g(x)}$
\[ P(X | \text{Accepted}) \]
\[ = \frac{P(X, \text{Accepted})}{P(\text{Accepted})} \]
\[ = \frac{P(\text{Accepted} | X) P(X)}{P(\text{Accepted})} \]
\[ = \frac{P(\text{Accepted} | X) P(X)}{P(\text{Accepted})} \]
\[ = \frac{f(x)}{\frac{f(x)}{M \cdot g(x)}} = \frac{f(x)}{M \cdot g(x)} \]
\[ \Pr \left( U < \frac{f(x)}{M \cdot g(x)} \mid x \right) \]
\[ \frac{f(x)}{M \cdot g(x)} \]
\[ \Pr(\text{Accepted}) = \sum_x \Pr(\text{Accepted} \mid X = x) \Pr(X \in x) \]

\[ \Pr(U < \frac{f(x)}{M \cdot g(x)} \mid X = x) \Rightarrow g(x) \]

\[ \frac{f(x)}{M \cdot g(x)} \]

\[ = \sum_x \frac{f(x)}{M} = \frac{1}{M} ? \]
Inverse Transformation

Goal: Sample from $f(x)$ (CDF $F(x)$)

Algo: Sample $U \sim U[0,1]$

\[ \text{return } F^{-1}(U) \]

Ability to compute $F^{-1}$
- Sampling Techniques
  - Transformation Based
    - Inversion Method
  - Rejection Based
  - Importance Sampling

- Usually beneficial in high dimensions.

\[
\frac{\pi r^2}{4 r^2} = \frac{\pi}{4}
\]
Box-Muller Method/Transform

\[ X_1 = R \cos \Theta \]
\[ X_2 = R \sin \Theta \]

\[ X_1, X_2 \sim^{\text{i.i.d.}} N(0,1) \]

\[ \Theta \sim U[0, 2\pi] \]
\[ R^2 \sim \text{Exp}(\frac{1}{2}) \]

Algorithm: Generate \( Y_1, Y_2 \sim^{\text{i.i.d.}} U[0, 1] \)

\[ R = \sqrt{-2 \log(Y_1)} \]
\[ \Theta = 2\pi Y_2 \]
\[ X_1 = R \cos \Theta \]
\[ X_2 = R \sin \Theta \]
Composition:

\[ F(x) = \sum_{i=1}^{\infty} p_i \cdot F_i(x) \]

\[ f(x) = \frac{1}{\sum_{i=1}^{n} \alpha_i \lambda_i} e^{-\lambda_i x} \]

\[ f(x) = \sum_{i=1}^{n} p_i \cdot f_i(x) \]

\[ 0 \leq p_i \leq 1 \]

\[ \sum_{i=1}^{n} p_i = 1 \]

\[ x_1, x_2, \ldots, x_n \]

\[ f_j, p_1, p_2, \ldots, p_n \]
Given:
Sample \((f_i) \forall i\)

\[\text{Algorithm.}\]
1) Generate \(I\) such that \(\Pr(I = i) = P_i \forall i\)
2) Generate a sample from \(F_{\sum_{i=1}^{N} r_i(I = i)}\) call it \(Y\)

\[\Pr(Y = k) = \sum_{i=1}^{N} \frac{r_i(I = i)}{P_i} F_i(Y) = F(x)\]