THE STRUCTURE OF THE QUASI ORDERED SETS OF $\aleph_1$-DENSE REAL ORDER TYPES WITH THE EMBEDDABILITY RELATION

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Thesis submitted in partial fulfillment of the requirements for the degree of Masters of Sciences

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2008

PREFACE

Let $A \subseteq \Re$. A is $\aleph_1$-*dense* if it has no first and no last element, and if between any two members of A there are exactly $\aleph_1$ members of A. We let K denote the collection of $\aleph_1$-dense subsets of $\Re$.

It is well known that every two countable dense subsets of $\Re$ without first and last element are order isomorphic. Baumgartner [B] proved that it is consistent with ZFC that every two $\aleph_1$-dense subsets of $\Re$ are order isomorphic.

The statement that each two $\aleph_1$ dense subsets of $\Re$ are order isomorphic is called *Baumgartner's Axiom*. This axiom is denoted by *BA*. In a universe which satisfies CH, for example, $\Re$, and $\Re \backslash \mathbb{Q}$ are two $\aleph_1$-dense sets which are not order isomorphic, thus BA is independent of ZFC.

This theorem of Baumgartner called for a variety of other questions concerning the structure of the class of $\aleph_1$-dense subsets of $\Re$ which were investigated by Abraham, Rubin and Shelah in [ARS]. Their paper introduced several techniques which were used to solve consistency questions concerning the properties of the embeddability relation on K. One of the solved question by [ARS] is the following.

Definition: An $\aleph_1$-dense set $A \subseteq \Re$ is homogeneous if for every $a, b \in A$, there is an automorphism f of $\langle A, \leq \rangle$ such that $f(a) = b$. The set of homogeneous $\aleph_1$-dense sets in $\Re$ is denoted by K^H.

Consider the structure $\left\langle K^H / \cong, \preceq \right\rangle$, where the members of $K^H / \cong$ are the equivalence classes of the relation of order isomorphism, and $\preceq$ is the order embeddability relation.

A question is asked whether, given a finite distributive lattice $\langle L, \leq \rangle$, there is a universe in which $\left\langle K^H / \cong, \preceq \right\rangle$, and $\langle L, \leq \rangle$ are isomorphic. In a universe which satisfies BA, for example, $\left\langle K^H / \cong, \preceq \right\rangle$ is isomorphic to a lattice with a single element. The answer, by [ARS], comes in the following theorem.

Define the following operation on $\left\langle K^H / \cong, \preceq \right\rangle$:

$$(A / \cong)^* = \{-a \mid a \in A\} / \cong$$

Then, $*$ is an automorphism of $\left\langle K^H / \cong, \preceq \right\rangle$, and indeed, it is an involution; that is, an automorphism of order ≤ 2.

Theorem: Let $\langle L, \leq, * \rangle$ be a finite poset with an involution. Then the following are equivalent:

(A) $MA_{\aleph_1} + \left(\left\langle K^H \cup \{\emptyset\} / \cong, \preceq, * \right\rangle \cong \langle L, \leq, * \rangle\right)$ is consistent with ZFC.

(B) $\langle L, \leq, * \rangle$ is a distributive lattice with an involution.

The proof in [ARS] is very intricate; some of the details are missing, and some of the claims are without explicit proof. In this thesis I will present a complete and detailed proof. I will also present some techniques used in [ARS], The main one is the so-called *club method*, which is used in a proof of the consistency of MA+BA.

Outline

Section 1 contains definitions of the structures K and K^H, and states some properties of sets in K and K^H under the assumption of $MA_{\aleph_1}$. *Section 2* is devoted to the Isomorphism forcings, which is a type of forcings that makes two sets of K order isomorphic. In this section we will also present some techniques such as the club-method, which serve as a core to the entire work. In *section 3* we state the main theorem and start its long proof; the iterated forcing which is used for this proof is outlined in *section 4*, along to a solution for a complicity which rises when entwining this forcing with $MA_{\aleph_1}$ iterated forcing. In this section we use the Explicit Contradiction Method technique. Finally, *section 5* is devoted entirely to the proof of the successor stage of this forcing.

Acknowledgment

I would like to thank Professor Matatyahu Rubin, my supervisor, for his many advices and constant support during this research. I am also thankful to Professor Uri Abraham for his guidance and support through the last months and the final stages of this research.

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May 2008

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1 Preliminaries

In this section we define the collection of $\aleph_1$-dense subsets of $\Re$, known as K, and discuss important aspects of K and K^H - the collection of all homogeneous $\aleph_1$-dense subsets of $\Re$. We will also show how to find non-isomorphic sets of K.

1.1 Discussing K and K^H under the assumption of $MA_{\aleph_1}$

Let $A \subseteq \Re$. A is $\aleph_1$-*dense* if it has no first and no last element, and between any two members of A, there are exactly $\aleph_1$ members of A.

Let $<$ be the order on $\Re$. If $A, B \subseteq \Re$, let $A \cong B$ mean that the structures $\langle A, < \rangle$ and $\langle B, < \rangle$ are isomorphic.

Let $A \preceq B$ mean that $\langle A, < \rangle$ is embeddable in $\langle B, < \rangle$. Let $A \perp B$ mean that there is no $\aleph_1$- dense set, $C \subseteq \Re$, such that $C \preceq A$, and $C \preceq B$; we will say that A is *far* from B. If this is not the case then we will say that A and B are *near*.

If $A \subseteq \Re$, let $A^* = \{-a \mid a \in A\}$.

For $A, B \subseteq \Re$, let $A \perp\!\!\!\perp B$ mean $A \perp B$, and $A \perp B^*$; we will say that A is *very far* from B.

An *order-preserving (OP)* function from A to B is a 1-1 function $f : A \to B$ such that for every $a_1, a_2 \in A$, $a_1 < a_2 \Rightarrow f(a_1) < f(a_2)$; it is *order-reversing (OR)* if it is 1-1, and for every $a_1, a_2 \in A$, $a_1 < a_2 \Rightarrow f(a_2) < f(a_1)$. If f is either OP, or OR, then f is called a *monotonic* function. It is easy to see that a composition of monotonic function with odd number of OR function is an OR function, and a composition of monotonic functions with even number of OR function is an OP function. In particular, a composition of OP function gives an OP function.

Let $K = \{A \subseteq \Re \mid A \text{ is } \aleph_1\text{-dense}\}$. In this work we will be interested in this collection, and even more in the collection K^H which will be defined later.

For the rest of this work let $MA_{\aleph_1}$ denote Martin's Axiom for c.c.c posets and

$\aleph_1$ dense sets. (see [K] II 2.5)

The following claim will give us a notion of the nature of K.

Claim 1.1. *(AC)*

Let $A \subseteq \Re$ be of the size of $\aleph_1$. Then A has a subset which is isomorphic to some $C \in K$.

Proof. Let $A \subseteq \Re$, $|A| = \aleph_1$. For every $a_1, a_2 \in A$ where $a_1 < a_2$, denote $a_1 \equiv_R a_2$ if $|A \cap (a_1, a_2)| = \aleph_0$. Clearly, $\equiv_R$ is an equivalent relation on A. As $|A/ \equiv_R | = \aleph_1$, then taking a representative from every equivalent set will ensure us a set B of the size of $\aleph_1$. Take $b_1, b_2 \in B$, where $b_1 \neq b_2$, and let $B' \stackrel{def}{=} (b_1, b_2) \cap B$. Every $b_3 < b_4$ in B' are in different equivalent sets, hence $B' \cap (b_3, b_4)$ is of the size of $\aleph_1$, and obviously B' has no first and last element; hence $B' \in K$. $\square$

In 1973 Baumgartner [B] proved the following.

Theorem 1.2. *It is consistent with ZFC that every two members of K are order isomorphic.*

Let BA be the following axiom: "Every two members of K are order isomorphic". A proof of the consistency of BA will be denoted here as BA *forcing* or the *Isomorphism forcing*, and in which the club method is used. More on that in section 2.

We now elaborate a discussion from section 6 in [ARS]. The following lemma is due to [ARS] 6.1.

Lemma 1.3. *Assume $MA_{\aleph_1}$. Then:*

1. *Let $A, B \in K$, and let $\{g_i \,|\, i \in \omega\}$ be a family of OP functions from A to B, such that for every $a \in A$ and $b_1, b_2 \in B$, $b_1 < b_2$, there is $i \in \omega$ such that $g_i(a) \in B \cap (b_1, b_2)$. Then $A \preceq B$.*

2. *Let A, B, and $\{g_i \,|\, i \in \omega\}$ be as in (1), and suppose in addition that for every $b \in B$ and $a_1, a_2 \in A$, $a_1 < a_2$, there is $i \in \omega$ such that $g_i^{-1}(b) \in A \cap (a_1, a_2)$. Then $A \cong B$.*

Proof. (2) We will define a c.c.c poset P as follows: $P = \{f : A \to B \mid f$ is finite, OP, and for every $a \in Dom(f)$ there is g_i such that $\langle a, f(a)\rangle \in g_i\}$. The ordering on P is defined by $f \leq g$ if $f \subseteq g$.

For every $\langle a_1, b_1\rangle, \langle a_2, b_2\rangle \in A \times B$, we say that $\{\langle a_1, b_1\rangle, \langle a_2, b_2\rangle\}$ is OP if $a_1 \leq a_2 \leftrightarrow b_1 \leq b_2$. We will call $\{\langle a_1, b_1\rangle, \langle a_2, b_2\rangle\}$ OR if $a_1 \leq a_2 \leftrightarrow b_2 \leq b_1$.

First notice, that for every $a \in A$, $D_a \stackrel{def}{=} \{f \in P \mid a \in Dom(f)\}$ is dense in P, and for every $b \in B$, $D^b \stackrel{def}{=} \{f \in P \mid b \in Rng(f)\}$ is dense in P. Hence if G is a P-generic filter which intersects all D_a's and D^b's, then $\bigcup\{f \mid f \in G\}$ is an isomorphism between $\langle A, <\rangle$ and $\langle B, <\rangle$. It remains to show that P is c.c.c.

Let $F \subseteq P$ be such that $|F| = \aleph_1$. We may assume that there is $n < \omega$ such that for every $f \in F$, $|f| = n$. For every $\alpha < \aleph_1$, and $i < n$ let $\langle a_i(\alpha), b_i(\alpha)\rangle$ denote the i'th member of f_α.

We can also assume that F is a Δ-system with an empty kernel. Hence, and due to the fact that for every $f \in F$, and $a \in Dom(f)$, $f(a) \in \{g_i(a) \mid i < \omega\}$, which is a countable collection, and the g_i's are all 1-1, we may assume that for every $\alpha, \beta < \aleph_1$, $Dom(f_\alpha) \cap Dom(f_\beta) = \emptyset$, and that $Rng(f_\alpha) \cap Rng(f_\beta) = \emptyset$. Hence

$f_\alpha \cup f_\beta$ is finite, 1-1, and for every $a \in f_\alpha \cup f_\beta$ there is $i < \omega$ such that $f(a) = g_i(a)$. It remains only to see that $f_\alpha \cup f_\beta$ is OP.

We can assume that there are $p_0 < \cdots < p_n$, $q_0 < \cdots < q_n$ rationals, such that for every $f_\alpha \in F$, $p_i < a_i(\alpha) < p_{i+1}$, and $q_i < b_i(\alpha) < q_{i+1}$. Hence, for every $\alpha, \beta < \aleph_1$, and $i, j < n$, $i \neq j$, $\{\langle a_i(\alpha), b_i(\alpha)\rangle, \langle a_j(\beta), b_j(\beta)\rangle\}$ is OP. As $\{g_i \mid i \in \omega\}$ is a countable collection, we can assume that for every $i < n$ there is such g that for every $f \in F$, $\langle a_i, b_i\rangle \in g$. Then for every $\alpha < \beta < \aleph_1$, $f_\alpha \cup f_\beta$ is OP, as for every $i < n$, $\{\langle a_i(\alpha), b_i(\alpha)\rangle, \langle a_i(\beta), b_i(\beta)\rangle\}$ is OP as well.

(1) Is done exactly the same. $\square$.

We now turn to look at the collection of homogeneous sets of K.

Definition 1.4. *Let $A \subseteq \Re$, and let $\leq$ be the order on $\Re$. $f : A \to A$ is an* automorphism *of $\langle A, <\rangle$, if f is bijective, and for every $a, b \in A$, $a \leq b \leftrightarrow f(a) \leq f(b)$.*
$A \subseteq \Re$ is homogeneous *if for every $a, b \in A$ there is an automorphism f of $\langle A, <\rangle$ such that $f(a) = b$.*
Let $K^H = \{A \in K \mid A$ is homogeneous$\}$.

Definition 1.5. *Let $A \in K$. $I \subseteq \Re$ is an* interval *of A, if there are $a_1, a_2 \in A$, $a_1 \neq a_2$, such that $I = (a_1, a_2) \cap A$.*

Note that if $A \in K$, and I is an interval of A, where $A \in K$, then $I \in K$ as well.

Definition 1.6. *For every $p < q < r < s \in Q$, if there are $a_1, a_2 \in A$ such that $a_1 \in (p, q)$, $a_2 \in (r, s)$, then let $I_{\{p,q,r,s\}}$ be an interval of A, such that if $a, a' \in A$, and $I_{\{p,q,r,s\}} = (a, a') \cap A$, then $a \in (p, q)$, and $a' \in (r, s)$. For $A \in K$, let ϕ^A be the following collection of intervals: $\phi^A = \{I_{\{p,q,r,s\}} \mid p, q, r, s \in \mathbb{Q}$, and $p < q < r < s\}$.*

Lemma 1.7. *Let τ be the topology of A which is defined by the collection of all intervals of A. Then ϕ^A is a countable base for τ.*

Proof. Let $A \in K$, and let $x \in A$. As A has no first and last element, there are $a_1, a_2 \in A$ such that $a_1 < x < a_2$. Let $p, q, r, s \in \mathbb{Q}$ where $p < q < x < r < s$, $a_1 \in (p, q)$, and $a_2 \in (r, s)$. Then there are $a, a' \in A$, such that $a \in (p, q)$, $a' \in (r, s)$ and $I_{\{p,q,r,s\}} \in \phi^A$, where $I_{\{p,q,r,s\}} \stackrel{def}{=} (a, a') \cap A$. Then $x \in I_{\{p,q,r,s\}}$.

It is also clear that for any $I_1, I_2 \in \phi^A$, if $x \in I_1 \cap I_2$, then there is $I_0 \in \phi^A$ such that $I_0 \subseteq I_1 \cap I_2$, and $x \in I_0$. Hence ϕ^A is a countable base for τ.

It is immediate from this lemma that every member of K is of the size of $\aleph_1$. It is also easy to see that every $A \in K$ is isomorphic to some $A' \in K$ which is dense in $\Re$. This is done by taking a member from every $I \in \phi^A$ thus forming a countable subset of A which is dense in itself - thus isomorphic to the rationals.

Lemma 1.8. *Let $A \in K^H$, $a \in A$, and let J be an interval of A. Then there is an interval I' of A, such that $I' \in \phi^A$, $a \in I'$, and an automorphism f of $\langle A, < \rangle$ such that $f(I') \subseteq J$.*

Proof. Let $b \in J$. As A is homogeneous, there is an automorphism f of $\langle A, < \rangle$ such that $f(a) = b$. As J has no first and last element, let $b_1, b_2 \in J$, $b_1 < b_2$, such that $b_1 < b < b_2$. Then there are $a_1 < a < a_2 \in A$ such that $f(a_1) = b_1$, and $f(a_2) = b_2$. Let $I = (a_1, a_2) \cap A$ be an interval of A. Then $f(I) \subseteq J$. As ϕ^A is a base, there is $I' \in \phi^A$, $I' \subseteq I$, $a \in I'$, then $f(I') \subseteq J$.

Lemma 1.9. *Assume $MA_{\aleph_1}$, and let $A \in K^H$. Then for every two intervals I_0, J_0 of A, $I_0 \cong J_0$.*

Proof. Let I_0, J_0 be two intervals of A. For every $J \in \phi^A$, $J \subseteq J_0$, let $\{\langle I_i^J, g_i^J\rangle \mid i \in \omega\}$ be such that $I_i^J \in \phi^A$, g_i^J is an automorphism of A, $g_i^J(I_i^J) \subseteq J$, and $\bigcup_{i\in\omega} I_i^J = A$. As ϕ^A is a countable base of A, then by lemma 1.8, there are such I_i^J's and g_i^J's.

let $\phi_0 = \{g_i^J \mid J \in \phi^A,\ J \subseteq J_0, \text{ and } i \in \omega\}$. Then I_0, J_0 and ϕ_0 satisfy the conditions of lemma 1.3 part (1). By a symmetric argument there is a family ϕ_1 such that J_0, I_0 and ϕ_1 also satisfy the conditions of lemma 1.3 part (1). Hence, I_0, J_0, and $\phi_0 \cup \{g^{-1} \mid g \in \phi_1\}$ satisfy the conditions of lemma 1.3 part (2), hence $I_0 \cong J_0$. $\square$

Let $G^A \overset{def}{=} \{g_{\{I,J\}} \mid I, J \in \phi^A$ and $g_{I,J}$ is an isomorphism between I, and $J\}$. We will denote G^A as the set of isomorphisms of $\langle A, <\rangle$ induced by ϕ^A.

We will now come to talk about *shuffles* of members of K, and K^H.

Definition 1.10. *Let $\{A_i \mid i < \alpha < \omega\} \subseteq K$, and let $B \in K$. We say that B is a* shuffle *of $\{A_i \mid i < \alpha\}$ if there are A_i^1, $i < \alpha$, such that $A_i^1 \cong A_i$, $B = \bigcup_{i<\alpha} A_i^1$, and for every $i < \alpha$ and $b_1, b_2 \in B$ where $b_1 < b_2$, there is $a \in A_i^1$ such that $b_1 < a < b_2$.*

Let $A \in K$; $B \in K$ is a mixing *of A if for every rational interval I there is $A_I \in K$ such that $A_I \subseteq I$, $A_I \cong A$, and $B = \bigcup\{A_I \mid I$ is a rational interval$\}$. B will be denoted as A^m.*

The following is lemma 6.1 from [ARS]. The proof that we give is a detailed elaboration of the proof given there.

Theorem 1.11. *Assume $(MA_{\aleph_1})$. Then:*

1. *If $A, B \in K^H$, $A \preceq B$, and $B \preceq A$, then $A \cong B$. (Hence $\preceq$ is a partial ordering of $K^H/\cong$).*

2. *If $A \in K^H$, then A is isomorphic to every non-empty open interval of A.*

3. *Let $\{A_i \mid i < \alpha \leq \omega\} \subseteq K^H$. Then (a) all shuffles of $\{A_i \mid i < \alpha\}$ are isomorphic and belong to K^H. (b) Suppose $C \in K^H$; if B is a shuffle of $\{A_i \mid i < \alpha\}$, and for every $i < \alpha$, $A_i \preceq C$, then $B \preceq C$. In particular, if all the A_i's are isomorphic to some fixed A, then every shuffle of $\{A_i \mid i < \alpha\}$ is isomorphic to A.*

4. *If $A \in K$ and B_1, B_2 are mixing of A, then $B_1 \cong B_2$, and $B_1 \in K^H$. If $C \in K^H$ and $A \preceq C$, then $B_1 \preceq C$. In particular if $A \in K^H$, then $A \cong B_1$.*

5. *If $A \in K$, and for every $B \in K$, $A \preceq B$, then $A \in K^H$.*

6. *If for every $A, B \in K$ $A \preceq B$, then BA holds.*

7. *If $|K^H/\cong| = 1$, then BA holds.*

Proof. Recall that for $A \in K$, G^A is the set of automorphisms of $\langle A, < \rangle$ induced by ϕ^A.

1. Let $f : A \to B$, and $g : B \to A$ be OP functions. Let $P = \{h : A \to B) \mid h$ is finite, OP, and if $\langle a, b \rangle \in h$, then there is $g' \in G^B$ such that $\langle f(a), b \rangle \in g'$, or there is $f' \in G^A$ such that $\langle a, g(b) \rangle \in f'\}$. As done before, P is c.c.c and if G is a generic filter over P then $\bigcup\{f \mid f \in G\}$ is an onto and OP function from A to B.

2. Let I be an interval of A. Then there is $G' \subseteq G^A$, such that for every $a \in A$, and $b_1, b_2 \in I$ where $b_1 < b_2$, there is $g' \in G'$ such that $g'(a) \in A \cap (b_1, b_2)$. Clearly G' is countable. So by 1.3 (1), $A \preceq I$. It is obvious that $I \preceq A$, so by 1.3 (1), $A \cong I$.

3. Let B_0, B_1 be shuffles of $\{A_i \mid i < \alpha\}$. Let $b \in B_0$, and $b_1, b_2 \in B_1$, $b_1 < b_2$. Then there is $i < \alpha$ such that $b \in A_i$. As A_i is dense in B_1, there are

$a_1 < a_2 \in A_i$, such that $b_1 \leq a_1 < a_2 \leq b_2$, and there is $g \in G^{A_i}$ such that $g(b) \in (a_1, a_2)$. So B, B', and $\bigcup_{i<\alpha} G^{A_i}$, meet the conditions of 1.3 (1), and as the case is symmetrical, the conditions of case 1.3 (2) are met, hence $B \cong B'$.

To see that $B \in K^H$, for every $a_0, b_0 \in B$, Let $P = \{f : B \to B \mid f$ is finite, OP, $\langle a_0, b_0 \rangle \in f$ and for every other $\langle a, b \rangle \in f$ there is $i < \alpha$ and $g \in G^{A_i}$ such that $\langle a, b \rangle \in g\}$. As $\bigcup_{i<\alpha} G^{A_i}$ is a countable collection, P is c.c.c and if G is a generic filter over P, then $\bigcup\{f \mid f \in G\}$ is an automorphism of B which puts a_0 in b_0. Hence, $B \in K^H$.

Let $C \in K^H$ be such that for every $i < \alpha$ $A_i \preceq C$. Let $f_i : A_i \to C$ be OP. Then for every $b \in B$ there is $i < \alpha$ such that $b \in A_i$, and for every $c_1, c_2 \in C \cap (\bigcup_{i<\alpha} Rng(f_i))$ there is $g \in G^C$ such that $g(f_i(b)) \in C \cap (c_1, c_2)$. So the conditions of 1.3 (1) are met and $B \preceq C$. Suppose there is $A \in K$ such that for every $i < \alpha$ $A_i \cong A$. Then $B \preceq A$. As $A \preceq A_i$, then $A \preceq B$, hence by (1) $A \cong B$.

4. Let B_1, B_2 be mixing of A. for every rational interval I, let $A_I, A'_I \subseteq I$, such that $A_I \cong A$, $A'_I \cong A$, and such that $\bigcup\{A_I \mid I$ is a rational interval$\} = B_1$, and $\bigcup\{A'_I \mid I$ is a rational interval$\} = B_2$.

   For every rational interval I, let $f_I : A_I \to A$ be an OP function, let $f'_I : A'_I \to A$ be an OP function, let $F = \{f_I \mid I$ is a rational interval$\} \cup \{f_I^{-1} \mid I$ is a rational interval$\}$, and let $F' = \{f_I \mid I$ is a rational interval$\} \cup \{f_I^{-1} \mid I$ is a rational interval$\}$. Recall that a composition of OP functions in an OP function.

   Let $a \in B_1$, and $b, b' \in B_2$, $b < b'$. Then there is a rational interval I, such that $a \in A_I$, and there is a rational interval J such that $J \subseteq (b, b')$. Then ${f'_J}^{-1}(f_I(a)) \in (b, b')$. So conditions 1.3 (1) are met, and because of symmetric reasons, conditions 1.3 (2) are met, hence $B_1 \cong B_2$.

We will show that $B_1 \in K^H$. Let $a_0, b_0 \in B_1$, and Let $P = \{f : B_1 \to B_1 \mid f$ is finite, OP, $\langle a_0, b_0 \rangle \in f$ and for every other $\langle a, b \rangle \in f$ there is $i < \alpha$ and $g, h \in F$ such that $\langle a, b \rangle \in g^{-}1 \circ h\}$. As before P is c.c.c and if G is a generic filter over B_1 then $\bigcup\{f \mid f \in G\}$ is an automorphism of B_1 which puts a in b. Hence, $B_1 \in K^H$.

Let $C \in K^H$ such that $A \preceq C$. let $f : A \to C$ be a 1-1 and OP function. Let $b \in B$, and let $c_1 < c_2 \in C \cap Rng(f)$. There is a rational interval I, such that $b \in A_I$, so let $f_I : A_I \to A$, and let G^A be the set of automorphisms of A induced by ϕ^A, then there is $g \in G^A$ such that $(f \circ g \circ f_I)(a) \in (c_1, c_2)$. so conditions 1.3 (1) are met so $B \preceq C \cap Rng(f)$, hence $B \preceq C$.

Suppose $C = A$, then $B \preceq A$, and as $B = \bigcup A_I$, where $A \cong A_I$, then $A \preceq B$; thus, by (1), $A \cong B$.

5. Let $A \in K$ be such that for every $B \in K$ $A \preceq B$.

   For every $I \in \phi^A$, as $I \in K$, then $A \preceq I$ and every two intervals of A are isomorphic, so G^A is defined well. Now for $a_0, b_0 \in A$, let $P = \{f : A \to A) | f$ is finite, OP, $\langle a_0, b_0 \rangle \in f$ and for every $\langle a, b \rangle \in f$ there is $g \in G^A$ such that $\langle a, b \rangle \in g\}$. Then as before P is c.c.c. Hence if G is a generic filter over A, then $\bigcup\{f \mid f \in G\}$ is an automorphism of A which puts a in b. Hence, $A \in K^H$.

6. If for every $A, B \in K$, $A \preceq B$, then by (5) $A, B \in K^H$, and then by (1) $A \cong B$. Thus for every $A, B \in K$, $A \cong B$ so BA holds.

7. Suppose $|K^H/\cong| = 1$. We will see that for every $A, B \in K$, $A \preceq B$; then by (6), BA holds.

For every $A \in K$ let A^m be a mixing of A. Let $A, B \in K$, and let B_j be a non empty interval of B. Then by (4) $A^m, B_j^m \in K^H$. As $|K^H/\cong| = 1$, $A^m \cong B_j^m$, and $A \preceq A^m$. So there is an OP function $g_{B_j} : A \to B_j^m$. By definition of B_j^m, let $B_j^m = \bigcup_{i\in\omega} B_j^i$, where for every i, $B^i \cong B_j$. Let $g_i : B^i \to B_j$ be the isomorphism between the two sets. Let $h_{B_j,i} = g_i^{-1} \circ g_{B_j}$. Then $h_{B_j,i} : A \to B_j$ is an OP function. As A, B, and $\{h_{B_j,i} \,|\, i,j < \omega\}$ meet case 1.3 (1), $A \preceq B$, as required. $\square$1.11

We will end this section with a surprising result of lemma 1.3 given by [ARS]

Definition 1.12. *For $A \in K^H$, we say that A is* n-homogeneous *if for every $a_1, \cdots, a_n, b_1, \cdots, b_n \in A$ such that $a_1 < \cdots < a_n$, and $b_1 < \cdots < b_n$, there is an automorphism f of $\langle A, <\rangle$ such that $f(a_1) = b_1, \cdots, f(a_n) = b_n$.*

Note that, if $A \in K^H$ is 2-homogeneous, then every two intervals of A are isomorphic.

Theorem 1.13. *Assume $MA_{\aleph_1}$, and let $n < \omega$. Then every $A \in K^H$ is n-homogeneous.*

Proof. For every $a_1 < \cdots < a_n, b_1 < \cdots < b_n \in A$, let P be the following poset: $P = \{f : A \to B \mid f$ is finite, OP, $\langle a_1, b_1\rangle, \cdots, \langle a_n, b_n\rangle \in f$, and for every other $\langle a, a'\rangle \in f$ there is $g \in G'$ such that $\langle a, a'\rangle \in G\}$.

As in lemma 1.3, P is c.c.c, and for every $a \in A$, $D_a = \{f \mid a \in Dom(f)\}$, and $D^a = \{f \mid a \in Rng(f)\}$ are dense. Hence, assuming $MA_{\aleph_1}$, there is generic filter over P, G, such that $\bigcup\{f \mid f \in G\}$ is an automorphism on A, that puts a_1 in b_1 , $\cdots$, a_n in b_n. $\square$

1.2 Finding non-isomorphic sets of K

We use lemma 9.9 from [ARS] as an example of a way to find sets of K which are not necessarily isomorphic. This lemma is presented in a way that will suit the needs of this work.

Lemma 1.14. *(CH) For $\kappa \leq \aleph_0$ let $\{A'_i \mid i < \kappa\}$ be a collection of sets of K (not necessarily disjoint), and let $\lambda < 2^{\aleph_0}$. For every $i < \lambda$ let $A_i \in K$, where $A_i = \{a(i,\alpha) \mid \alpha < \aleph_1\}$ is a 1-1 standard enumeration of A_i. Then there are pairwise disjoint $\{B'_i \mid i < k\} \subseteq K$ such that:*

1. *For every $i < k$, $B'_i \subseteq A'_i$, and B'_i is dense in A'_i.*
2. *For every $i < k$, $B'_i \perp (B'_i)^*$, and for every $i \neq j$, $B'_i \perp\!\!\!\perp B'_j$.*
3. *There is a club $C \subseteq \aleph_1$, such that for every $i < \lambda$, let $B_i = \{a(i,\alpha) \mid \alpha \in C\}$ be a subset of A_i; then for each $i < \lambda$, $B_i \in K$ and is dense in A_i, and for every $i < \lambda$, $j < \omega$, $B_i \perp\!\!\!\perp B'_j$.*

Proof. For every $i < \kappa$ let $\{a(\lambda + i, \alpha) \mid \alpha < \aleph_1\}$ be a 1-1 standard enumeration of A'_i. let $h : (\lambda + \kappa) \times \aleph_1 \to \aleph_1$ be defined as $h(i,\alpha) = a(i,\alpha)$. Let $R = \{\langle i, \alpha, \beta\rangle \mid a(i,\alpha) < a(i,\beta)\}$, and let $M = \langle \lambda + \kappa, <, h, R\rangle$. We can also assume, for later use, that M contains an enumeration of the rational intervals.

For any $N \prec M$, if $|N| \cap \aleph_1 = \alpha$, denote N as N_α. Let $C_M \overset{def}{=} \{\alpha \mid N_\alpha \prec M\}$. C_M is closed unbounded, and will be denoted through the entire work as the club of the elementary substructures of M. For every $\alpha, \beta \in C_M$ where $\alpha < \beta$, $[\alpha, \beta)$ will be denoted as a C_M-slice. Note that C_M forms a partition over $\aleph_1$, which is the collection of C_M-slices denoted as E^C; let $\{E^C_\alpha \mid \alpha < \aleph_1\}$ be its enumeration in an increasing order, where $\{\alpha \mid \alpha < min(C)\}$ is regarded as E^C_0.

Let $D \subseteq \aleph_1$ be a club such that $|\aleph_1 \backslash D| = \aleph_1$. Let $\{D'_i \mid i < \kappa\}$ be a partition of $|\aleph_1 \backslash D|$, where $|D'_i| = \aleph_1$ for every $i < \kappa$. Let $C = \bigcup\{E^C_\alpha \mid \alpha \in D\}$, and for every

$i < \lambda$, let $B_i = \{a(i, \alpha) \mid \alpha \in C\}$. Clearly, C is a club. We will see that B_i is a member of K and dense in A_i.

For every $a_1, a_2 \in B_i$ where $a_1 < a_2$, let $a_1 = a(i, \alpha_1)$, $a_2 = a(i, \alpha_2)$, and let $N_0 \prec M$ be a model such that $h(i, \alpha_1), h(i, \alpha_2) \in N_0$ (meaning $a_1, a_2 \in N_0$). Let $\gamma = N_0 \cap \aleph_1$. As $A \in K$, the following sentence is valid in M:

$$\varphi \equiv (\forall x \exists y > x)(h(i, \alpha_1) < h(i, y) < h(i, \alpha_2)).$$

Let $j : \omega_1 \to \omega_1$ be an increasing function. For every C^M slice E^C_α, let $N_{j(\alpha)}$ denote the elementary substructure of M such that E^C_α is the last C^M slice in it. so φ is valid in every $N_{j(\alpha)}$, where $\alpha > \gamma$. As such, in every $N_{j(\alpha)}$, there are unbounded many β's, that for each β $h(i, \alpha_1) < h(i, \beta) < h(i, \alpha_2)$. Hence there is $\beta \in E^C_\alpha$, such that $a(i, \beta) \in B_i$, and $a_1 < a(i, \beta) < a_2$. As there are uncountable many such C^M slices, (a_1, a_2) contains $\aleph_1$ members of B_i. Same argument also holds to show that B_i has no first or last element, hence $B_i \in K$, and as this argument holds for any $a_1 < a_2$ in A, B_i is dense in A_i as well.

Now for each $i < \kappa$, we will find $B'_i \subseteq A'_i$ such that the B'_i's are members of K, pairwise disjoint, and have the following properties:

1. If $a(\lambda + \kappa, \alpha) \in B'_i$, then $\alpha \in \bigcup\{E^C_\gamma \mid \gamma \in D'_i\}$.
2. If $a(\lambda + \kappa, \alpha), a(\lambda + \kappa, \beta) \in B'_i$ and are distinct, then $E^C(a(\lambda + \kappa, \alpha)) \neq E^C(a(\lambda + \kappa, \beta))$.
3. For every $i < \kappa$, $B'_i \in K$ and dense in A'_i.

We define $\{B'_i \mid i < \kappa\}$ by induction on $\aleph_1$. Let ϕ be the countable base for the topology on $\bigcup_{i<\kappa} A'_i$ (see Definition 1.6). Suppose $\phi = \{I_j \mid j < \omega\}$. For $i < \kappa$ and a limit ordinal δ, let $B'_i(\delta) = \bigcup_{\alpha<\delta} B'_i(\alpha)$. Suppose we have defined $B'_i(\alpha)$, and

$B'_l(\alpha+1)$ for every $l<i$. For $B'_i(\alpha+1)$ we pick an element a_j from every interval I_j, $j<\omega$, where $I_j \cap A'_i \neq \emptyset$, such that a_j has not been picked before, and such that if $a_j = a(\lambda+i,\beta)$, then $\beta \in E^C_\gamma$, where $\gamma \in D'_i$, and no other element from E^C_γ was yet picked. As so far we have picked only a countable elements from every I_j, and as D'_i is unbounded, we are free to pick such a_j. For every $i<\kappa$ we define $B'_i = \bigcup_{\alpha<\aleph_1} B'_i(\alpha)$. Then the B'_i's are pairwise disjoint and satisfy the required conditions.

Suppose by contradiction that for $i<\lambda$, $j<\kappa$, $f \subseteq B'_j \times B_i$ is an uncountable monotonic function (either OP or OR). Let F be the closure of f. For simplicity we will treat B'_j, B_i as elements of $\aleph_1$, where $\leq$ will be the ordinal order on $\aleph_1$, and $\preceq$ will denote the linear order on $\aleph_1$ induced by the reals. We will see that for every $a,b<\aleph_1$, $|F(a)| \leq 3$, and $|F^{-1}(b)| \leq 3$. Let $\langle a,b\rangle \in f$, and suppose that for some $b \prec b_0$, there is $\langle a,b_0\rangle \in F$. We will see that there is no $\langle a,b_1\rangle \in F$ such that $b \prec b_0 \prec b_1$.

Suppose there was such $\langle a,b_1\rangle \in F$. Then as $\langle a,b_0\rangle \in F$, there is $\langle a'_0,b'_0\rangle \in f$ such that $b \prec b'_0 \prec b_1$, and as f is OP then $a \prec a'_0$. Now as $\langle a,b_1\rangle \in F$, there is a series $\{\langle a^i,b^i\rangle \in f \mid i<\omega\}$ that converges to $\langle a,b_1\rangle$; So start with a certain $i<\omega$, for $j>i$, $b'_0 \prec b^j$. so as f is OP, then it is also that $a'_0 \prec a^j$, a contradiction, as then $\{\langle a^i,b^i\rangle \in f \mid i<\omega\}$ does not converge to $\langle a,b_1\rangle$.

Hence there is only one point in F of the type $\langle a,b_0\rangle$ such that $b \prec b_0$. From symmetrical reasons, there is only one point in F of the type $\langle a,b_1\rangle$ such that $b_1 \prec b$. So $|F(a)| \leq 3$. In the same way it shows that $|F^{-1}(b)| \leq 3$ for any $b<\aleph_1$. If f is an OR function then it is shown in the very same way.

Now ,assuming M contains all rational intervals as well, let $d \in |M|$ be such that F is definable from d. Let $\gamma_0 < \aleph_1$ be such that for every $\gamma \geq \gamma_0$, if $\gamma \in C$, then there is $N \prec M$ such that $d \in |N|$ and $|N| \cap \aleph_1 = \gamma$. Let $\langle a,b\rangle \in f$, where

$a = h(i, \alpha)$, $b = h(\lambda + j, \beta)$, and $\gamma_0 \leq \alpha, \beta$. We can assume that $\alpha \leq \beta$ as it will not matter. By the definition of B'_j and C, α and β are in different E^C slices, so there is $\gamma \in C$ such that $\alpha < \gamma \leq \beta$. Let $N \prec M$ be such that $d \in |N|$ and $|N| \cap \aleph_1 = \gamma$. Hence $\alpha \in |N|$, where $\beta \notin |N|$, and F is definable from d in $|N|$. Now, as $|F(\alpha)| \leq 3$, then all the elements $b < \aleph_1$ for which $F(\alpha) = b$ are in $|N|$, a contradiction; as β is such an element and $\beta \notin |N|$. Thus for every $i < \lambda$, $j < \kappa$, $B'_j \perp\!\!\!\perp B'_i$.

As for every $i, j \in \kappa$, $i \neq j$, and for every $a \in B'_j$, $b \in B'_i$ where $a = h(\lambda + j, \alpha)$ and $b = h(\lambda + i, \beta)$, α and β are in separated E^C slices, the same argument shows that $B'_i \perp\!\!\!\perp B'_j$. Last, as for every $i \in \kappa$, and $a, b \in B'_i$ where $a = h(\lambda + i, \alpha)$ and $b = h(\lambda+i, \beta)$, α and β are also in a separated E^C slices, we use the same argument to show that $B'_i \perp (B'_i)^*$. $\square$

2 Isomorphism forcings

The following is mostly chapter 9 from [ARS] In which the Isomorphism forcing (also known as the BA-forcing) is discussed. The intricate details of the Isomorphism forcing are essential for this work. The theorems which are discussed here are theorems 9.1, 9.2, and lemma 9.6, all from [ARS].

The axiom BA is "Every two members of K are order isomorphic". Baumgartner [B] showed that BA is consistent with ZFC, and the way of proof in [B] also showed that BA is consistent with $MA_{\aleph_1}$. This suggested that maybe $MA_{\aleph_1}$ already implies BA. The negative answer to this question was found by Shelah. Using two techniques he invented: the *club method*, which is explained in this section, and the *explicit contradiction method* (see section 4.2), Shelah [AS] proved that $MA_{\aleph_1}$ is consistent with an entangled set (as defined in [AS]), thus showing that $MA_{\aleph_1}$ does not imply BA. Abraham [AS] then found another way to refute BA, by using the club method to construct a universe which satisfies $MA_{\aleph_1}$, and in which there is a set $A \in K$, such that A is not isomorphic to A^*.

In this section we will use the club method to prove that BA is consistent with ZFC. The proof is the same proof of Theorem 9.1 from [ARS], to which we have added an elaborated explanation of the club method. Theorem 2.3 (Theorem 9.2 from [ARS]) is a variation of 2.1, dealing with order isomorphism between shuffles of sets of K. Lemma 2.4 (lemma 9.6 from [ARS]), denoted as the *Isomorphism-Farness Lemma*, is another variation of 2.1, and it shows how to force two members of K to become isomorphic, while keeping other members of K far from each other. This lemma will be followed by a discussion which will show some results which are developed from the Isomorphism-Farness Lemma.

Note that in the following, CH is assumed, and as a result of that, the continuum is $\aleph_2$ in a universe which satisfies BA. However section 5 of [ARS] shows how to satisfy BA with the continuum enlarged beyond $\aleph_2$. A different axiom, denoted as $A1$, is assumed as a substitute of CH. In this work I have chosen to stay with the assumption of CH.

2.1 The Isomorphism forcing

The Isomorphism forcing, or the BA-forcing, is an $\aleph_2$-stages finite support iteration of c.c.c forcing sets. We will focus on the successor stage of the BA-forcing, without explaining the whole $\aleph_2$ iteration; these are well known, and done by the method of Solovey and Tenenbaum [ST]. Hence, given $A, B \in K$, we construct a forcing set $P_{A,B}$ which makes A, and B isomorphic.

Let $\mathcal{E}$ be a partition of $\aleph_1$, and $\sigma \subseteq \aleph_1 \times \aleph_1$. We define the graph $G^{\mathcal{E}}_{\sigma}$. The set of vertices of the graph $V^{\mathcal{E}}_{\sigma}$ is $\{E \in \mathcal{E} \mid (\exists a, b)(\langle a, b\rangle \in \sigma, \text{ and } (a \in E \text{ or } b \in E))\}$. The set of edges is σ, and E_1, E_2 which belong to $V^{\mathcal{E}}_{\sigma}$ are connected by $\langle a, b\rangle \in \sigma$, if $a \in E_1$, and $b \in E_2$ or $b \in E_1$, and $a \in E_2$. When $\mathcal{E}$ is fixed and σ varies, we denote $G^{\mathcal{E}}_{\sigma}$, and $V^{\mathcal{E}}_{\sigma}$ by G_{σ}, and V_{σ} respectively.

We say that a graph G is *cycle free* if it does not contain cycles, i.e. it does not contain a sequence of vertices $a_1, \cdots, a_n$ and a set of distinct edges $e_1, \cdots, e_n$ such that e_i connects a_i and a_{i+1}, and e_n connects a_n and a_1. Let $C \subseteq \aleph_1$ be a club, let $\mathcal{E}^C$ denote the set of C-slices (that is, the partition induced by C on $\aleph_1$), and let $\{E^C_i \mid i < \aleph_1\}$ be an enumeration of $\mathcal{E}^C$ in an increasing order. We regard the set $E = \{\alpha \mid \alpha < min(C)\}$ as a C-slice, hence $E = E^C_0$. E^C_i, and E^C_j are *near*, if for some $n \in \omega$, $i + n = j$, or $j + n = i$.

Let $\alpha \in \aleph_1$. $E^C(\alpha)$ denotes the member of $\mathcal{E}^C$ to which α belongs. $E^C(\alpha)$ is abbreviated by $E(\alpha)$ when C is fixed.

Let $\prec$ be a linear ordering of a subset of $\aleph_1$, $C \subseteq \aleph_1$ be a club, and $A, B \subseteq \aleph_1$. We define $P = P(C, \prec, A, B)$:

$P = \{f : A \to B \mid f$ is a finite OP function with respect to $\prec$, G_f^C is cycle free, and if $f(a) = b$ then $E^C(a)$, and $E^C(b)$ are near$\}$.

$f \leq g$ if $f \subseteq g$.

Theorem 2.1. *(CH)*

Let $A, B \in K$, and M be a model such that $|M| \supseteq \aleph_1$. Assume that there is a linear ordering $\prec$ on $\aleph_1$ definable in M such that $\langle \aleph_1, \preceq \rangle$ is embeddable in $\langle \Re, < \rangle$, and $\langle A \cup B, < \rangle$ is embeddable in $\langle \aleph_1, \prec \rangle$. We can assume that $A, B \subseteq \aleph_1$, and $\aleph_1$ and the usual linear ordering $<$ of $\aleph_1$ are definable in M. Let C be the club of elementary substructures of M, and suppose further that for every C-slice, E, $\langle A \cap E, \prec \rangle$ and $\langle B \cap E, \prec \rangle$ are dense in $\langle A, \prec \rangle$ and $\langle B, \prec \rangle$ respectively. Then: (a) $P = P(C, \prec, A, B)$ is c.c.c; and (b) $\Vdash_P A \cong B$.

Proof. (b) Let $f \in P$ and let $a \in A \backslash Dom(f)$. We show that there is $g \geq f$ such that $a \in Dom(g)$.

V_f is finite, hence there is a C-slice E such that $E \notin V_f$, $E \neq E(a)$, and E and $E(a)$ are near. Since B is dense in itself and $B \cap E$ is dense in B, there is $b \in B \cap E$ such that $g \stackrel{def}{=} f \cup \{\langle a, b \rangle\}$ is OP. By the choice of E, $g \in P$, hence g is as required. Similarly if $b \in B \backslash Rng(f)$, then there is $g \in P$ such that $b \in Rng(g)$. Now let G be a P-generic filter. Then $\bigcup G$ is an onto, and OP function from A into B. That proves (b).

(a) If V and W are sets of pairs of real numbers we say that $\langle V, W \rangle$ is OP if for every $\langle v_0, v_1 \rangle \in V$, and $\langle w_0, w_1 \rangle \in W$, $\{\langle v_0, v_1 \rangle, \langle w_0, w_1 \rangle\}$ is OP, that is $v_0 < w_0 \leftrightarrow v_1 < w_1$. Analogously we define the notion $\langle V, W \rangle$ is order reversing (OR). Note that if U_i, $i = 0, \cdots, 3$, are pairwise disjoint intervals, then $\langle U_0 \times U_1, U_2 \times U_3 \rangle$ is either OP or OR.

Let $\{f_\alpha \mid \alpha < \aleph_1\} = F_0 \subseteq P$. We uniformize F_0 as much as possible. We thus assume that there is $n \in \omega$ such that for every $f \in F_0$, $|f| = n$. We can assume that F_0 is a Δ-system, and that it suffices to deal with the case when the kernel of F_0 is empty. Hence let us assume that

$$f_\alpha = \{\langle a(\alpha,0), a(\alpha,1)\rangle, \cdots, \langle a(\alpha,2n-2), a(\alpha,2n-1)\rangle\}$$

where the $a(\alpha,2i)$'s are distinct, and if $\alpha < \beta$, $c \in \mathit{Dom}(f_\alpha) \cup \mathit{Rng}(f_\alpha)$ and $d \in \mathit{Dom}(f_\beta) \cup \mathit{Rng}(f_\beta)$, then $E(c) \neq E(d)$. This condition assures us that if $f_\alpha \cup f_\beta$ is OP, then $f_\alpha \cup f_\beta \in P$.

Last, we can assume that there are $q_0, \cdots, q_n, r_0, \cdots, r_n \in \mathbb{Q}$, such that $q_0 < , \cdots, < q_n$, $r_0 <, \cdots, < r_n$, and for every $f_\alpha \in F_0$, and $i < n$, $q_i < a(\alpha, 2i) < q_{i+1}$, and $r_i < a(\alpha, 2i+1) < r_{i+1}$. This condition assures us that for every $f_\alpha, f_\beta \in F_0$, if $i, j < n$, $i \neq j$, then $\{\langle a(\alpha,2i), a(\alpha,2i+1)\rangle, \langle a(\beta,2j), a(\beta,2j+1)\rangle\}$ is OP, hence we need only to find two conditions $f_\beta, f_\gamma \in F_0$, such that for every $i < n$, $\{\langle a(\beta,2i), a(\beta,2i+1)\rangle, \langle a(\gamma,2i), a(\gamma,2i+1)\rangle\}$ is OP; then $f_\beta \cup f_\gamma$ is OP.

Let $a(\alpha) = \langle a(\alpha,0), \cdots, a(\alpha,2n-1)\rangle$, $F_1 = \{a(\alpha) \mid \alpha < \aleph_1\}$, and let F be the topological closure of F_1 in $(\langle \aleph_1, \prec\rangle)^{2n}$. It will be convenient (however not necessary) to assume that all the $a(\alpha,i)$'s are distinct, hence we assume that $A \cap B = \emptyset$.

Let $D \in |M|$ be such that F is definable from D in M, and there is some countable open base of $\langle \aleph_1, \prec\rangle$ consisting of intervals whose elements are definable from D in M. Let $\gamma_0 \in C$ be such that $C \cap [\gamma_0, \aleph_1) \subseteq \{\alpha \mid (\exists N \prec M)(D \in |N|$ and $|N| \cap \aleph_1 = \alpha\}$. Let $f_\alpha = f$ be such that for every $i < 2n$, $\gamma_0 \leq a(\alpha, i)$. We denote $a(\alpha, i)$ by $a(i)$, $a(\alpha) = a$, and $W = \mathit{Dom}(f_\alpha) \cup \mathit{Rng}(f_\alpha)$.

Let $E_0, \cdots, E_{k-1}$ be the set of C-slices which intersect W, arranged in an increasing order. Let $x = \langle x(0), \cdots, x(2n-1)\rangle$ be a sequence of variables. For every $s < k$, let $R_s = \{i \mid a(i) \in E_s\}$, $a_s = a \restriction R_s$ and $x_s = x \restriction R_S$. Hence $\bigcup_{s<k} a_s = a$ and $\bigcup_{s<k} x_s = x$.

We are now ready for the duplication argument. We define by a downward induction on $s = k, \cdots, 0$ formulas $\varphi_s(x_0, \cdots, x_{s-1})$ such that $M \models \varphi_s[a_0, \cdots, a_{s-1}]$.

Let

$$\varphi_k \equiv \bigcup_{s<k} x_s \in F.$$

Then $M \models \varphi_k[a_0, \cdots, a_{k-1}]$.

Suppose φ_{s+1} has been defined, and we define φ_s.

Let $Q_x\varphi(x)$ mean: "there are unboundedly many pairwise disjoint x's satisfying $\varphi(x)$". For a set S, and an element t, let $B(S,t)$ mean: "for every $s \in S$, $t < s$".

Using the fact that $E_0, \cdots, E_{k-1}$ are arranged in an increasing order, there is $\delta \in C$ such that for every $i \in R_{s-1}$, and $j \in R_s$, $a(i) < \delta < a(j)$. Let $N_\delta \prec M$ be such that $|N_\delta| \cap \aleph_1 = \delta$. Then for every $b \in N_\delta$,

$$M \models (\exists x_s)B(x_s, b) \wedge \varphi_{s+1}(a_0, \cdots, a_{s-1}, x_s).$$

.

As $N_\delta \prec M$, then for every $b \in N_\delta$

$$N_\delta \models (\exists x_s)B(x_s, b) \wedge \varphi_{s+1}(a_0, \cdots, a_{s-1}, x_s).$$

.

Hence,

$$N_\delta \models Q(x_s)\varphi_{s+1}(a_0, \cdots, a_{s-1}, x_s),$$

and again, as $N_\delta \prec M$

$$M \models Q(x_s)\varphi_{s+1}(a_0, \cdots, a_{s-1}, x_s)$$

.

So the set $A_s = \{a_s \mid M \models \varphi_{s+1}(a_0, \cdots, a_{s-1}, a_s)\}$ is unbounded, hence there are $a_s^0, a_s^1 \in A_s$, such that $a_s^0 \cap a_s^1 = \emptyset$, and

$$M \models (\exists x_s^0, x_s^1) \left(Rng(x_s^0) \cap Rng(x_s^1) = \emptyset \wedge \bigwedge_{l=0}^{1} \varphi_{s+1}(a_0, \cdots, a_{s-1}, x_s^l) \right)$$

.

For every $i \in R_s$ and $l = 0, 1$ let U_i^l be an interval definable in D such that: (1) the U_i^l's are pairwise disjoint; (2) if $s' > s$, $i' \in R_{s'}$, and $l' \in \{0, 1\}$, then $U_i^l \cap U_{i'}^{l'} = \emptyset$; and (3)

$$M \models (\exists x_s^0, x_s^1) \left(\bigwedge_{l=0}^{1} \left(x_s^l \in \prod_{i \in R_s} U_i^l \right) \wedge \bigwedge_{l=0}^{1} \varphi_{s+1}(a_0, \cdots, a_{s-1}, x_s^l) \right).$$

Let φ_s be the following formula:

$$\varphi_s \equiv (\exists x_s^0, x_s^1) \left(\bigwedge_{l=0}^{1} \left(x_s^l \in \prod_{i \in R_s} U_i^l \right) \wedge \bigwedge_{l=0}^{1} \varphi_{s+1}(a_0, \cdots, a_{s-1}, x_s^l) \right).$$

We thus proved $M \models \varphi_s$ for every $s < k$.

We prove that we can find for every $s < k$, $l(s) \in \{0, 1\}$ such that for every $i < n$; if $2i \in R_s$, and $2i + 1 \in R_t$, then $\left\langle U_{2i}^{l(s)} \times U_{2i+1}^{l(t)}, U_{2i}^{1-l(s)} \times U_{2i+1}^{1-l(t)} \right\rangle$ is OP.

Recall that: (1) The graph G_f has vertices $E_0, \cdots, E_{k-1}$. (2) The edges of G_f are $\langle a(0), a(1) \rangle, \cdots, \langle a(2n-2), a(2n-1) \rangle$. (3) E_s is connected to E_t by

$\langle a(2i), a(2i+1)\rangle$ if $a(2i) \in E_s$, and $a(2i+1) \in E_t$, or $a(2i) \in E_t$, and $a(2i+1) \in E_s$.
(4) G_f is cycle free.

Let $S \subseteq k$ be such that for every component T of G_f there is a unique $s \in S$ such that $E_s \in T$. Let $S_j = \{s \in k \mid$ there is $t \in S$ and a 1-1 path in G_f of length j connecting E_s with $E_t\}$.

Since G_f is cycle free, the S_j's are pairwise disjoint and moreover for every $s \in S_{j+1}$ there is a unique edge in G_f which connects E_t with some element of $\{E_s \mid s \in S\}$.

For every $s \in S_0$ define $l(s) = 0$. Suppose $l(s)$ has been defined for every $s \in \bigcup_{m \leq j} S_m$. Let $t \in S_{j+1}$; let $\langle a(2i), a(2i+1)\rangle$ be the unique edge connecting E_t to an element of $\{E_s \mid s \in S_j\}$, and without loss of generality suppose that $s \in S_j$, $a(2i) \in E_s$, and $a(2i+1) \in E_t$. Define $l(t)$ in such a way that $\left\langle U_{2i}^{l(s)} \times U_{2i+1}^{l(t)}, U_{2i}^{1-l(s)} \times U_{2i+1}^{1-l(t)} \right\rangle$ will be OP. We have defined $l(s)$ for every $s \in k$, and it is easy to check that $\{l(s) \mid s \in k\}$ is as required.

Using the φ_i's we will now construct two members of F. Since $M \models \varphi_0$, there is b_0^0 such that $M \models (b_0^0 \in \prod_{i \in R_0} U_i^{l(0)} \wedge \varphi_1(b_0^0))$. Suppose $b_0^0, \cdots, b_{s-1}^0$ have been defined in such way that $M \models \varphi_s[b_0^0, \cdots, b_{s-1}^0]$; hence by the definition of φ_s, there is b_s^0 such that $M \models (b_s^0 \in \prod_{i \in R_s} U_i^{l(s)} \wedge \varphi_{s+1}(b_0^0, \cdots, b_s^0))$. According to this definition we obtain $b_0^0, \cdots, b_{k-1}^0$ such that $\bigcup_{s<k} b_s^0 \in F$ (this is assured by φ_k) and for every $s < k$, $b_s^0 \in \prod_{i \in R_s} U_i^{l(s)}$. Similarly we can define b_s^1, $s < k$, such that $\bigcup_{s<k} b_s^1 \in F$ and $b_s^1 \in \prod_{i \in R_s} U_i^{1-l(s)}$.

For $l = 0, 1$, let $b^l = \bigcup_{s<k} b_s^l$. So suppose $b^l = \left\langle b^l(0), \cdots, b^l(2n-1)\right\rangle$. By the construction of the $l(s)$'s, for every $i < n$, $\{\left\langle b^0(2i), b^0(2i+1)\right\rangle, \left\langle b^1(2i), b^1(2i+1)\right\rangle\}$ is OP.

Since $F = cl(\{a(\alpha) \mid \alpha < \aleph_1\})$, there are β, γ such that for every $s < k$,

$$a(\beta) \restriction R_s \in \prod_{i \in R_s} U_i^{l(s)} \text{ and } a(\gamma) \restriction R_s \in \prod_{i \in R_s} U_i^{1-l(s)}.$$

Thus for every $i < n$, $\{\langle a(\beta, 2i), a(\beta, 2i+1)\rangle, \langle a(\gamma, 2i), a(\gamma, 2i+1)\rangle\}$ is OP. Hence $f_\beta \cup f_\gamma$ is OP, and by the uniformization its graph is cycle free, so $f_\beta \cup f_\gamma \in P$, so P is c.c.c. $\square$

We now turn to discuss isomorphism between shuffles of sets of K. We will concentrate only on the finite cases, although the following can be easily expanded to $\aleph_0$, and even $\aleph_1$ cases.

Definition 2.2. *Let $\{A_i \mid i < n\}$, and $\{B_i \mid i < n\}$ be collections of sets of K , such that, for every $i < n$, A_i is dense in $\bigcup_{i<n} A_i$, and B_i is dense in $\bigcup_{i<n} B_i$.*

We say that $\bigcup_{i<n} A_i$, and $\bigcup_{i<n} B_i$ are shuffle-isomorphic *if there is an onto OP function $f : \bigcup_{i<n} A_i \to \bigcup_{i<n} B_i$, such that for every $i < n$, $f(A_i) = B_i$. Let $f_i \overset{def}{=} f \restriction A_i$ be the order isomorphism from A_i to B_i induced by f.*

We will prove the following

Theorem 2.3. *(CH)*

Let $A = \{A_i \mid i < n\}$, and $B = \{B_i \mid i < n\}$ be as in Definition 2.2. Then there is an $\aleph_1$ c.c.c forcing set P, such that $\Vdash_P$ (A and B are shuffle isomorphic).

Proof. This proof is very similar to the proof of Theorem 2.1. We start from a model M which universe is $\aleph_1$, and which encodes A, B, and all the relevant information. As before, we let C be the club of elementary substructures of M. We define $P = \{f : A \to B \,|\, f$ is a finite OP function with respect to $\prec$, G^C_f is cycle free, if $f(a) = b$ then $E^C(a)$, and $E^C(b)$ are near, and for every $i < n$, $f(A_i) \subseteq B_i$, and $f^{-1}(B_i) \subseteq A_i\}$. $f \leq g$ if $f \subseteq g$.

The proof that P is c.c.c is identical to the proof of Theorem 2.1. It is left to see that forcing with P make A and B shuffle isomorphic.

For every $b \in A$ the set $D_b \stackrel{def}{=} \{f \in P \mid b \in Dom(f)\}$ is dense, as for every $i < n$, A_i is dense in A; and so is $D^b \stackrel{def}{=} \{f \in P \mid b \in Rng(f)\}$. Hence, if G is a P-generic filter, then $\bigcup G$ is an onto OP function from A to B where for every $i < n$, $f(A_i) = B_i$. $\Box$

2.2 The Isomorphism-Farness Lemma

Suppose there are $A, B, G, H \in K$ where $G \perp H$, and we want to find a forcing set which will make A and B isomorphic, but G and H will continue to remain far from each other. Lemma 2.4 states certain conditions which will insure that G, and H remain far. We denote this lemma as the *Isomorphism-Farness Lemma*. The discussion continues at section 2.3. We use Theorem 2.1 to define the required forcing set; however, we will have to work harder in order to show the farness is kept.

The following is a slightly less powerful version of lemma 9.6 from [ARS] (see remark 2.5).

Lemma 2.4. *(CH) Let $\gamma < 2^{\aleph_0}$. For every $i < \gamma$, let $G_i, H_i \in K$ such that $G_i \perp H_i$. Let $A, B \in K$ be such that for every $i < \delta$, $A \perp\!\!\!\perp G_i$, and $B \perp\!\!\!\perp H_i$. Then there is a c.c.c forcing set P of power $\aleph_1$ such that $\Vdash_P A \cong B$, and for every $i < \gamma$, $\Vdash_P G_i \perp H_i$.*

Remark 2.5. *A set $A \in K$ is* increasing *if there is no uncountable OR function g with no fixed points such that $Dom(g), Rng(g) \subseteq A$. A is* 2-entangled *if there is no such uncountable OP function as well.*

The original lemma 9.6 from [ARS] also claimed that if G_i is increasing then $\Vdash_P$" G_i is increasing", and if G_i is 2-entangled then $\Vdash_P$" G_i is 2-entangled". These *claims are proved in a very similar way, as the following proof of this lemma.*

Another difference from lemma 9.6 from [ARS] is that assuming a certain axiom A_1 instead of CH, one can obtain the lemma for $\gamma < 2^{\aleph_1}$ (see [ARS] section 5 for more about A_1).

Proof. First we construct a model which encodes all the information we need. Let $h : A \cup B \to \aleph_1$ be a 1-1 function, and for every $i < \lambda$, let $h_i : A \cup B \cup G_i \cup H_i \to \aleph_1$ be a 1-1 onto function containing h. Let M be the following model: (1)$|M| = \aleph_1 \cup \lambda$. (2) M has a three-place relation $R \overset{def}{=} \{\langle i, \alpha, \beta \rangle \mid h_i^{-1}(\alpha) < h_i^{-1}(\beta)\}$. We denote $\alpha \prec_i \beta$ to mean that $\langle i, \alpha, \beta \rangle \in R$. (3) M has unary predicates which represent $h(A)$, and $h(B)$. (4) Finally, M has the binary relations $S_G = \{\langle i, \alpha \rangle \mid \alpha \in G_i\}$, and $S_H = \{\langle i, \alpha \rangle \mid \alpha \in H_i\}$.

Let $\alpha \prec^0 \beta$ denote that $h^{-1}(\alpha) < h^{-1}(\beta)$, hence $\prec^0$ is definable in M. Let C be the club of the elementary substructures of M, and let $P = P(C, \prec^0, A, B)$ be defined as in 2.1. By 2.1, P is c.c.c, and it isomorphizes A and B.

We next show that for every $i < \lambda$, $\Vdash_P G_i \perp H_i$. Suppose by contradiction there is $i < \lambda$, and $p \in P$, such that $p \Vdash_P \neg(G_i \perp H_i)$.

We denote $G = G_i$, $H = H_i$, and $\prec = \prec_i$. By abuse of notation we assume $A \cup B \cup G \cup H = \aleph_1$. Let τ be a P-name such that $p \Vdash_P$"τ is an uncountable OP function and $\tau \subseteq G \times H$". We can assume that $p = 0$.

Let $\{\langle f_\alpha, \langle a_\alpha, b_\alpha \rangle \rangle \mid \alpha < \aleph_1\}$ be such that for every α, $f_\alpha \Vdash_P \langle a_\alpha, b_\alpha \rangle \in \tau$, and if $\alpha \neq \beta$ then $\langle a_\alpha, b_\alpha \rangle \neq \langle a_\beta, b_\beta \rangle$.

We will reach a contradiction if we find α and β such that $f_\alpha \cup f_\beta \in P$, but $\{\langle a_\alpha, b_\alpha \rangle, \langle a_\beta, b_\beta \rangle\}$ is not OP. We uniformize $\{\langle f_\alpha, \langle a_\alpha, b_\alpha \rangle \rangle \mid \alpha < \aleph_1\}$ as in 2.1, hence we denote $f_\alpha = \{\langle a(\alpha, 0), a(\alpha, 1) \rangle, \cdots, \langle a(\alpha, 2n-2), a(\alpha, 2n-1) \rangle\}$. We can assume that all the $a(\alpha, i)$'s are distinct. We also denote $a_\alpha = a(\alpha, 2n)$ and $b_\alpha = a(\alpha, 2n+1)$.

Let $a(\alpha) = \langle a(\alpha,0),\cdots,a(\alpha,2n+1)\rangle$, $F_1 = \{a(\alpha) \mid \alpha < \aleph_1\}$ and let F be the closure of F_1 in $(\langle\aleph_1,\prec\rangle)^{2n+2}$. We define D, γ_0, $a(i)$, a, W, etc, as in 2.1.

In the duplication argument we distinguish between two cases.

case 1. $E(a(2n)) = E(a(2n+1))$. Let v be such that $E(a(2n)) = E_v$. We define φ_s inductively as in 2.1, except for the case where $s = v$. Suppose φ_{v+1} has been defined. Recall that $Q_x\varphi(x)$ mean: "there are unboundedly many pairwise disjoint x's satisfying φ".

$$M \models Q(x_v)\,((x(2n) \in G) \wedge (x(2n+1) \in H) \wedge \varphi_{v+1}(a_0,\cdots,a_{v-1},x_v))\,.$$

Since $G \perp H$,

$M \models (\exists x_v^0, x_v^1)(Rng(x_v^0) \cap Rng(x_v^1) = \emptyset \wedge \bigwedge_{l=0}^1 (x_{2n}^l \in G) \wedge \bigwedge_{l=0}^1 (x_{2n+1}^l \in H)$
$\wedge(\{\langle x_{2n}^0, x_{2n+1}^0, \rangle, \langle x_{2n}^1, x_{2n+1}^1, \rangle\}$ is OR) $\wedge \bigwedge_{l=0}^1 \varphi_{v+1}(a_0,\cdots,a_{v-1},x_v^l))$.

For every $i \in R_v$, and $l = 0,1$, let U_i^l be an interval definable from D such that all the U_i^l's so far defined are pairwise disjoint and

$M \models (\exists x_v^0, x_v^1)(\bigwedge_{l=0}^1 x_v^l \in \prod_{i\in R_v} U_i^l \wedge$
$\{\langle x_{2n}^0, x_{2n+1}^0, \rangle, \langle x_{2n}^1, x_{2n+1}^1, \rangle\}$ is OR $\wedge \bigwedge_{l=0}^1 \varphi_{v+1}(a_0,\cdots,a_{v-1},x_v^l))$.

Let $\varphi_v(x_0,\cdots,x_{v-1})$ be the formula obtained from the above formula by substituting a_s by x_s for every $s < v$.

As in 2.1, we can find for every $s < k$, $l(s) \in \{0,1\}$ such that for every $i < n$: if $2i \in R_s$, and $2i+1 \in R_t$, then $\left\langle U_{2i}^{l(s)} \times U_{2i+1}^{l(t)}, U_{2i}^{1-l(s)} \times U_{2i+1}^{1-l(t)}\right\rangle$ is OP. We continue as in 2.1 and find β,γ such that for every $s < k$

$a(\beta) \restriction R_s \in \prod_{i\in R_s} U_i^{l(s)}$ and $a(\gamma) \restriction R_s \in \prod_{i\in R_s} U_i^{1-l(s)}$.

It follows that $f_\beta \cup f_\gamma \in P$. Since $\langle a_\beta, b_\beta\rangle \in U_{2n}^{l(v)} \times U_{2n+1}^{l(v)}$, and $\langle a_\gamma, b_\gamma\rangle \in U_{2n}^{1-l(v)} \times U_{2n+1}^{1-l(v)}$, it follows that $\{\langle a_\beta, b_\beta\rangle, \langle a_\gamma, b_\gamma\rangle\}$ is OR. A contradiction.

case 2. $E(a(2n)) \neq E(a(2n{+}1))$. Let $E(a(2n)) = E_v$, and $E(a(2n{+}1)) = E_w$.

case 2.1: E_v and E_w are not in the same component of G_f. In this case we define φ_s, $s \leq k$, exactly as in 2.1. Let $S_0 \subseteq k$ be a set such that $v, w \in S_0$, and for every component L of G_f, $|S_0 \cap \{s \,|\, E_s \in L\}| = 1$. We define S_i, as in 2.1. Next we define $l(s)$ for every $s \in S_0$. For every $s \in S_0 \backslash \{w\}$, let $l(s) = 0$. We define $l(w)$ to be equal to 0 or 1 according to whether $\left\langle U^0_{2n} \times U^0_{2n+1}, U^1_{2n} \times U^1_{2n+1} \right\rangle$ is OR or OP. We now define $l(s)$ for $s \in S_i$ by induction on i as in 2.1. Let $\beta, \gamma < \aleph_1$ be such that for every $s < k$

$a(\beta) \restriction R_s \in \prod_{i \in R_s} U^{l(s)}_i$ and $a(\gamma) \restriction R_s \in \prod_{i \in R_s} U^{1-l(s)}_i$.

It it easy to see that $f_\beta \cup f_\gamma \in P$, and that $\{\langle a_\beta, b_\beta \rangle, \langle a_\gamma, b_\gamma \rangle\}$ is OR. A contradiction.

case 2.2: E_v and E_w are in the same component of G_f. Let $v = v_0, v_1, \cdots, v_r = w$ be such that $E_{v_0}, \cdots, E_{v_r}$, is the unique path in G_f connecting E_v and E_w. By the symmetry between the roles of A and B we can assume that E_{v_0} and E_{v_1} are connected by $\langle a(2j), a(2j+1) \rangle$ where $a(2j) \in E_{v_0}$ and $a(2j+1) \in E_{v_1}$. We define φ_s for $s < k$ inductively. φ_k is defined as in 2.1. If $s \neq v$, then φ_s is defined from φ_{s+1} as in 2.1. Suppose φ_{v+1} has been defined, and we define φ_v; then

$$M \models Q(x_v)(\varphi_{v+1}(a_0, \cdots, a_{v-1}, x_v) \wedge x_{2j} \in A \wedge x_{2n} \in G).$$

Since $A \perp\!\!\!\perp G$,

$M \models (\exists x^{0,P}_v, x^{1,P}_v)(\wedge^1_{l=0} \varphi_{v+1}(a_0, \cdots, a_{v-1}, x^{l,P}_v) \wedge (\{\left\langle x^{0,P}_{2j}, x^{0,P}_{2n} \right\rangle, \left\langle x^{1,P}_{2j}, x^{1,P}_{2n} \right\rangle\}$ is OP)), and

$M \models (\exists x^{0,R}_v, x^{1,R}_v)(\wedge^1_{l=0} \varphi_{v+1}(a_0, \cdots, a_{v-1}, x^{l,R}_v) \wedge (\{\left\langle x^{0,R}_{2j}, x^{0,R}_{2n} \right\rangle, \left\langle x^{1,R}_{2j}, x^{1,R}_{2n} \right\rangle\}$ is OR)).

Let $U_i^{l,P}, U_i^{l,R}$, $l = 0, 1$, and $i \in R_v$ be pairwise disjoint open intervals, disjoint from previously defined l's, definable from D such that: (1) $\left\langle U_{2j}^{0,P} \times U_{2n}^{0,P}, U_{2j}^{1,P} \times U_{2n}^{1,P} \right\rangle$ is OP, and $\left\langle U_{2j}^{0,R} \times U_{2n}^{0,R}, U_{2j}^{1,R} \times U_{2n}^{1,R} \right\rangle$ is OR; and (2)

$$M \models \bigwedge_{l=0}^{1} \exists x_v^{l,P} \left(x_v^{l,P} \in \prod_{i \in R_v} U_i^{l,P} \wedge \varphi_{v+1}(a_0, \cdots, a_{v-1}, x_v^{l,P}) \right)$$
$$\wedge \bigwedge_{l=0}^{1} \exists x_v^{l,R} \left(x_v^{l,R} \in \prod_{i \in R_v} U_i^{l,R} \wedge \varphi_{v+1}(a_0, \cdots, a_{v-1}, x_v^{l,R}) \right).$$

Let φ_v be the formula obtained from the above formula by replacing each a_s, $s < v$, by x_s. This concludes the definition of the φ_s's.

Our next goal is to define $l(s)$ for every $s < k$. In fact we also have to decide whether to use the $U_i^{l,P}$'s or the $U_i^{l,R}$'s. Let $T = \{s \mid E_s$ and E_v are connected in $G_f\}$. We define $l(s)$ for $s \in k \backslash T$ as in 2.1. Let Z be defined as follows. If $\left\langle U_{2j+1}^{l(v_1)} \times U_{2n+1}^{l(w)}, U_{2j+1}^{1-l(v_1)} \times U_{2n+1}^{1-l(w)} \right\rangle$ is OP, then $Z = R$, and if the above pair of sets is OR, then $Z = P$. We denote each $U_i^{l,Z}$ by U_i^l and proceed in the definition of $l(s)$ as in 2.1. Let β and γ be such that for every $s < k$

$$a(\beta) \restriction R_s \in \prod_{i \in R_s} U_i^{l(s)} \text{ and } a(\gamma) \restriction R_s \in \prod_{i \in R_s} U_i^{1-l(s)}.$$

By the proof of 2.1, $f_\beta \cup f_\gamma \in P$. We check that $\{\langle a_\beta, b_\beta \rangle, \langle a_\gamma, b_\gamma \rangle\}$ is OR.

By the construction of $l(s)$, $\left\langle U_{2j}^{l(v)} \times U_{2j+1}^{l(v_1)}, U_{2j}^{1-l(v)} \times U_{2j+1}^{1-l(v_1)} \right\rangle$ is OP.

$\left\langle U_{2j}^{l(v)} \times U_{2n}^{l(v)}, U_{2j}^{1-l(v)} \times U_{2n}^{1-l(v)} \right\rangle$ was chosen to be OR or OP according to whether $\left\langle U_{2j+1}^{l(v_1)} \times U_{2n+1}^{l(w)}, U_{2j+1}^{1-l(v_1)} \times U_{2n+1}^{1-l(w)} \right\rangle$ was OP or OR respectively. Since the composition of an OP and an OR function is an OR function, it follows that

$\left\langle U_{2n}^{l(v)} \times U_{2n+1}^{l(w)}, U_{2n}^{1-l(v)} \times U_{2n+1}^{1-l(w)} \right\rangle$ is OR.

Since $\langle a_\beta, b_\beta \rangle \in U_{2n}^{l(v)} \times U_{2n+1}^{l(w)}$, and $\langle a_\gamma, b_\gamma \rangle \in U_{2n}^{1-l(v)} \times U_{2n+1}^{1-l(w)}$, it follows that $\{\langle a_\beta, b_\beta \rangle, \langle a_\gamma, b_\gamma \rangle\}$ is OR. Hence we reach a contradiction again. $\square$

2.3 More about the Isomorphism-Farness Lemma

Lemma 2.4 has presented tools which can be used to handle questions regarding keeping farness of sets, while making other sets isomorphic. In this section we will look more closely at these tools, and try to present them in a way which will be easy to use later on. We will end this section with lemma 2.9 which is an example of how these tools can be used to elaborate the Isomorphism-Farness lemma. We start with a generalization of case 1 in lemma 2.4

Claim 2.6. *(CH) Suppose that there are $B, C \in K$, where $B \perp C$, and there are $A, A' \in K$, that we use BA-forcing (see theorem 2.1) to make A, and A' isomorphic. Suppose that in the process of the proof of theorem 2.1, E_s is an E-slice such that there are $a(i), a(j) \in E_s$, where $a(i) \in B$, and $a(j) \in C$.*

Then we can make the conditions $f_\beta, f_\gamma \in F_1$, that were found as a result of Theorem 2.1, and such that $f_\beta \cup f_\gamma \in P$, to have the ability that $\{\langle f(\beta,i), f(\beta,j)\rangle, \langle f(\gamma,i), f(\gamma,j)\rangle\}$ is OR.

Note that, as G_f is cycle free, and $a(i), a(j) \in E_s$, then $\langle a(i), (a(j)\rangle$, and $\langle a(j), (a(i)\rangle$ are NOT edges in G_f.

Proof. We start making A and A' isomorphic by 2.1. Suppose we are in the duplication process, had just defined φ_{s+1}, and are about to define φ_s. As $a(i), a(j) \in E_s$, it follows that $i, j \in R_s$. Suppose $a(i) \in B$, and $a(j) \in C$, then

$$M^c \models Q(x_s)(\varphi_{s+1}(a_0, \cdots, a_{v-1}, x_s) \wedge x_i \in B \wedge x_j \in C).$$

As $B \perp C$, there is no uncountable OP function between B and C, so

$$M^c \models (\exists x_s^0, x_s^1)\left(\bigwedge_{l=0,1} \varphi_{s+1}(a_0, \cdots, a_{s-1}, x_s^l) \wedge (\{\langle x_i^0, x_j^0\rangle, \langle x_i^1, x_j^1\rangle\} \text{ is OR})\right).$$

Hence we choose $\{U_t^l \mid l \in \{0,1\},\ t \in R_s\}$ pairwise disjoint open intervals, disjoint from the previously defined l's, definable from D, such that for the specific $i, j \in R_s$: $\left\langle U_i^0 \times U_j^0, U_i^1 \times U_j^1 \right\rangle$ is OR.

Then let φ_s be the following formula:

$$\varphi_s \equiv \bigwedge_{l=0,1} \exists x_s^l \left(x_s^l \in \prod_{t \in R_s} U_t^l \wedge \varphi_{s+1}(a_0, \cdots, a_{w-1}, x_s^l) \right).$$

When we define $l(t)$ for every $t < k$, it does not matter whether $l(s) = 0$, or $l(s) = 1$, as $\left\langle U_i^{l(s)} \times U_j^{l(s)}, U_i^{1-l(s)} \times U_j^{1-l(s)} \right\rangle$ is OR. Hence when choosing β and γ such that for every $t < k$

$$a(\beta) \restriction R_t \in \prod_{i \in R_t} U_i^{l(t)} \text{ and } a(\gamma) \restriction R_t \in \prod_{i \in R_t} U_i^{1-l(t)},$$

and picking $f_\beta, f_\gamma \in F_1$ close to $a(\beta), a(\gamma)$ respectively, then $f_\beta \cup f_\gamma \in P$, and $\{\langle f(\beta,i), f(\beta,j)\rangle, \langle f(\gamma,i), f(\gamma,j)\rangle\}$ is OR. $\square$.

If indeed we follow the above claim and choose such $\{U_t^l \mid l \in \{0,1\},\ t \in R_s\}$, such that for the specific $i, j \in R_s$: $\left\langle U_i^0 \times U_j^0, U_i^1 \times U_j^1 \right\rangle$ is OR, we say that $\{U_t^l | l \in \{0,1\}\ ,\ t \in R_s\}$ is $\{i,j\}$-*OR*.

Suppose now that $B, C \in K$, $B \perp C^*$, and we want to make some $A, A' \in K$ isomorphic. Suppose that $a(i) \in B$, $a(j) \in C$, and $a(i), a(j) \in E_s$. We follow the above claim, using the fact that $B \perp C^*$, and we can choose $\{U_t^l \mid l \in \{0,1\}\ ,\ t \in R_s\}$, such that for the specific $i, j \in R_s$: $\left\langle U_i^0 \times U_j^0, U_i^1 \times U_j^1 \right\rangle$ is OP. So we can find $f_\beta, f_\gamma \in F_1$, such that $f_\beta \cup f_\gamma \in P$, and $\{\langle f(\beta,i), f(\beta,j)\rangle, \langle f(\gamma,i), f(\gamma,j)\rangle\}$ is OP. In this case we will say that $\{U_t^l \mid l \in \{0,1\},\ t \in R_s\}$ is $\{i,j\}$-*OP*.

Now suppose that there are $B, C \in K$, $B \perp\!\!\!\perp C$, and we want to make some $A, A' \in K$ isomorphic. Suppose that $a(i) \in B$, $a(j) \in C$, and $a(i), a(j) \in E_s$. Then, as in case 2.2 of lemma 2.4, we can choose two sets of such open intervals. one set, $\{U_t^{l,P} \,|\, l \in \{0,1\},\, t \in R_s\}$ is $\{i,j\}$-OP, and another set $\{U_t^{l,R} \,|\, l \in \{0,1\},\, t \in R_s\}$ which is $\{i,j\}$-OR. Then we can choose conditions $f_\beta, f_\gamma \in F_1$ in which $f_\beta \cup f_\gamma \in P$, and $\{\langle f(\beta,i), f(\beta,j)\rangle, \langle f(\gamma,i), f(\gamma,j)\rangle\}$ is OP, and conditions $f_\delta, f_\epsilon \in F_1$ in which $f_\delta \cup f_\epsilon \in P$, and $\{\langle f(\delta,i), f(\delta,j)\rangle, \langle f(\epsilon,i), f(\epsilon,j)\rangle\}$ is OR.

Note that there are cases when we can make $f_\beta, f_\gamma \in F_1$ such that $f_\beta \cup f_\gamma \in P$, and such that for some i, j, $\{\langle f(\beta,i), f(\beta,j)\rangle, \langle f(\gamma,i), f(\gamma,j)\rangle\}$ is OR, and for some g, h, $\{\langle f(\beta,g), f(\beta,h)\rangle, \langle f(\gamma,g), f(\gamma,h)\rangle\}$ is OP (or OR as well). For example, if E_t is another E-slice, different from E_s, then there might be some $a(g), a(h) \in E_t$ such that there are $G, H \in K$, where $G \perp H$, and $a(g) \in G$, $a(h) \in H$. Then we can choose a set of open intervals $\{U_e^l \,|\, l \in \{0,1\},\, e \in R_s\}$ which is $\{i,j\}$-OR, and a set of open intervals $\{U_e^l \,|\, l \in \{0,1\},\, e \in R_t\}$ which is $\{g,h\}$-OR; then such f_β, f_γ can be created.

Suppose now that there are j, k, that are not necessarily in the same E-slice. We want to see when $\{\langle f(\beta,j), f(\beta,k)\rangle, \langle f(\gamma,j), f(\gamma,k)\rangle\}$ is OP.

Claim 2.7. *Let $\langle a_0, b_0\rangle$, $\langle a_1, b_1\rangle$, and $\langle c_0, c_1\rangle$ be pairs of elements of $\Re$. Then $\{\langle a_0, c_0\rangle, \langle a_1, c_1\rangle\}$ is OP if and only if $\{\langle a_0, b_0\rangle, \langle a_1, b_1\rangle\}$ is of the same nature (either OP, or OR) as $\{\langle b_0, c_0\rangle, \langle b_1, c_1\rangle\}$.*

Proof. Obvious.

We will say that $\{\langle a_0, c_0\rangle, \langle a_1, c_1\rangle\}$ is a *composition* of $\{\langle a_0, b_0\rangle, \langle a_1, b_1\rangle\}$, and $\{\langle b_0, c_0\rangle, \langle b_1, c_1\rangle\}$. It is easy to see that a composition of an odd number of OR pairs yields an OR pair, and a composition of an even number of OR pairs yields an OP pair.

Suppose now, we want to make some $A, A' \in K$ isomorphic via Theorem 2.1. In the proof, let $E_{v_0}, \cdots, E_{v_r}$ be the unique path in G_f, such that $a(j) \in E_{v_0}$, and $a(k) \in E_{v_r}$. For every $i < r$, let $\langle a(\eta_i), a(\mu_{i+1})\rangle$ denote the edge in G_f which connects $E(v_i)$ with $E(v_{i+1})$. Suppose $a(j) = a(\eta_0)$, and $a(k) = a(\mu_r)$.

Theorem 2.8. *1. Let S be a subset of $\{1, \cdots, r-1\}$. Suppose that for every $i \in S$, in creating φ_{v_i}, we can construct a set of intervals $\{U_t^l \mid l \in \{0,1\}, t \in R_{v_i}\}$ which is $\{\eta_i, \mu_i\}$-OR, and for every $i \notin S$, we can construct a set of intervals $\{U_t^l \mid l \in \{0,1\}, t \in R_{v_i}\}$ which is $\{\eta_i, \mu_i\}$-OP.*

Then if $|S|$ is even, we can make the conditions $f_\beta, f_\gamma \in F_1$, that were found as a result of Theorem 2.1, and such that $f_\beta \cup f_\gamma \in P$, to be such that

$\{\langle f(\beta,j), f(\beta,k)\rangle, \langle f(\gamma,j), f(\gamma,k)\rangle\}$ is OP,

and if $|S|$ is odd, we can make these conditions to be such that

$\{\langle f(\beta,j), f(\beta,k)\rangle, \langle f(\gamma,j), f(\gamma,k)\rangle\}$ is OR.

2. Suppose there is $i \in \{1, \cdots, r-1\}$ such that in creating φ_{v_i}, we can construct a set of intervals $\{U_t^{l,P} \mid l =\in \{0,1\}, t \in R_{v_i}\}$ which is $\{\eta_i, \mu_i\}$-OP, and a set of intervals $\{U_t^{l,R} \mid l \in \{0,1\}, t \in R_{v_i}\}$ which is $\{\eta_i, \mu_i\}$-OR.

Then we can choose conditions $f_\beta, f_\gamma \in F_1$ that were found as a result of Theorem 2.1, in which $f_\beta \cup f_\gamma \in P$, and

$\{\langle f(\beta,j), f(\beta,k)\rangle, \langle f(\gamma,j), f(\gamma,k)\rangle\}$ is OP, and we can choose such conditions $f_\delta, f_\epsilon \in F_1$ in which $f_\delta \cup f_\epsilon \in P$, and $\{\langle f(\delta,j), f(\delta,k)\rangle, \langle f(\epsilon,j), f(\epsilon,k)\rangle\}$ is OR.

Proof. 1) Suppose f_β, and f_γ were found by the above conditions. As the composition of even number of OR pairs is an OP pair, and the composition of odd number of OR pairs is an OR pair, then $\{\langle f(\beta,j), f(\beta,k)\rangle, \langle f(\gamma,j), f(\gamma,k)\rangle\}$ can be OP, if there is an even number of OR pairs which compose them.

Suppose $|S|$ is even, then as $f_\beta \cup f_\gamma$ are OP, then for every $i < r$, $\{\langle a(\beta,\eta_i), a(\beta,\mu_{i+1})\rangle, \langle a(\gamma,\eta_i), a(\gamma,\mu_{i+1})\rangle\}$ is OP.

Then the only pairs that can be OR are those of the sort $\{\langle a(\beta,\eta_i), a(\beta,\mu_i)\rangle$, $\langle a(\gamma,\eta_i), a(\gamma,\mu_i)\rangle\}$; and $\{i \mid \{\langle a(\beta,\eta_i), a(\beta,\mu_i)\rangle,\ \langle a(\gamma,\eta_i), a(\gamma,\mu_i)\rangle\}$ is OR $\} = S$, which is even, so $\{\langle f(\beta,j), f(\beta,k)\rangle, \langle f(\gamma,j), f(\gamma,k)\rangle\}$ is OP.

The case were $|S|$ is odd is proved in the same way.

2) Let f_β, f_γ, be the conditions that were found using the set of intervals which is $\{i,j\}$-OP, and let f_δ, f_ϵ be the conditions that were found using the set of intervals which is $\{i,j\}$-OR; we can make this to be the only difference in their construction.

Then $\{\langle a(\beta,\eta_0), a(\beta,\mu_i)\rangle,\ \langle a(\gamma,\eta_0), a(\gamma,\mu_i)\rangle\}$ is OP, if and only if $\{\langle a(\delta,\eta_0), a(\delta,\mu_i)\rangle$, $\langle a(\epsilon,\eta_0), a(\epsilon,\mu_i)\rangle\}$ is OP; same, $\{\langle a(\beta,\eta_i), a(\beta,\mu_r)\rangle,\ \langle a(\gamma,\eta_i), a(\gamma,\mu_r)\rangle\}$ is OP, if and only if $\{\langle a(\delta,\eta_i), a(\delta,\mu_r)\rangle,\ \langle a(\epsilon,\eta_i), a(\epsilon,\mu_r)\rangle\}$ is OP.

Hence, as the type of $\{\langle a(\beta,\eta_0), a(\beta,\mu_r)\rangle,\ \langle a(\gamma,\eta_0), a(\gamma,\mu_r)\rangle\}$ depends on the composition of $\{\langle a(\beta,\eta_0), a(\beta,\mu_i)\rangle,\ \langle a(\gamma,\eta_0), a(\gamma,\mu_i)\rangle\}$, $\{\langle a(\beta,\mu_i), a(\beta,\eta_i)\rangle,\ \langle a(\gamma,\mu_i), a(\gamma,\eta_i)\rangle\}$, and $\{\langle a(\beta,\eta_i), a(\beta,\mu_r)\rangle,\ \langle a(\gamma,\eta_i), a(\gamma,\mu_r)\rangle\}$,

and same goes for $\{\langle a(\delta,\eta_0), a(\delta,\mu_r)\rangle,\ \langle a(\epsilon,\eta_0), a(\epsilon,\mu_r)\rangle\}$,

then $\{\langle a(\beta,\eta_0), a(\beta,\mu_r)\rangle,\ \langle a(\gamma,\eta_0), a(\gamma,\mu_r)\rangle\}$ is OP if and only if $\{\langle a(\delta,\eta_0), a(\delta,\mu_r)\rangle$, $\langle a(\epsilon,\eta_0), a(\epsilon,\mu_r)\rangle\}$ is OR. $\square$

We can now expand lemma 2.4 and receive the following:

Lemma 2.9. *Let $\lambda < 2^{\aleph_1}$; for every $i < \lambda$ let $G_i, H_i \in K$ such that $G_i \perp H_i$. Let $A, B \in K$ such that for every $i < \lambda$, one of the following cases happens:*

***case 1**: $A \perp\!\!\!\perp G_i$ and $B \perp\!\!\!\perp G_i$.*

***case 2**: $A \perp A^*$, $B \perp B^*$, $A \perp B^*$,*

$A \perp H_i$, $B \perp H_i$, $A \perp (G_i)^*$, and $B \perp (G_i)^*$.

Then there is an $\aleph_1$ c.c.c forcing set P, such that $\Vdash_P A \cong B$, and such that for every $i < \lambda$, $\Vdash_P G_i \perp H_i$.

Proof. As this lemma is supplemental to lemma 2.4, we set the conditions as in 2.4, and define the same model M, and the same forcing set $P = P(C, \prec^0, A, B)$, which is mentioned there.

Then, as in 2.4, $\Vdash_P (A \cong B)$.

Suppose that there is $i < \lambda$, and $p \in P$ such that $p \Vdash_P \neg(G_i \perp H_i)$. We denote G_i as G, H_i as H,and $\prec$ as $\prec_i$. By abuse notation we assume $A \cup B \cup G \cup H = \aleph_1$. If G, H meet **case 1** of the lemma, that is, $A \perp\!\!\!\perp G$ and $B \perp\!\!\!\perp G$, or $A \perp\!\!\!\perp H$ and $B \perp\!\!\!\perp H$, then, by lemma 2.4, we reach a contradiction.

Then let us assume that G, and H meet **case 2** of the lemma. Let τ be a P-name such that $p \Vdash_P$"τ is an uncountable OP function and $\tau \subseteq G \times H$". We can assume that $p = 0$. Let $\{\langle f_\alpha, \langle a_\alpha, b_\alpha\rangle\rangle \mid \alpha < \aleph_1\}$ be such that for every α, $f_\alpha \Vdash_P \langle a_\alpha, b_\alpha\rangle \in \tau$, and if $\alpha \neq \beta$ then $\langle a_\alpha, b_\alpha\rangle \neq \langle a_\beta, b_\beta\rangle$. If we find $\alpha, \beta < \aleph_1$ such that $f_\alpha \cup f_\beta \in P$, but $\langle a_\alpha, b_\alpha\rangle$, $\langle a_\beta, b_\beta\rangle$ is not OP, we reach a contradiction. We uniformize $\{\langle f_\alpha, \langle a_\alpha, b_\alpha\rangle\rangle \mid \alpha < \aleph_1\}$ as in 2.4, hence we denote

$f_\alpha = \{\langle a(\alpha, 0), a(\alpha, 1)\rangle, \cdots, \langle a(\alpha, 2n-2), a(\alpha, 2n-1)\rangle\}$. We can assume that all the $a(\alpha, i)$'s are distinct. We also denote $a_\alpha = a(\alpha, 2n)$ and $b_\alpha = a(\alpha, 2n+1)$. Let $a(\alpha) = \langle a(\alpha, 0), \cdots, a(\alpha, 2n+1)\rangle$, $F_1 = \{a(\alpha) \mid \alpha < \aleph_1\}$ and let F be the closure of F_1 in $(\langle \aleph_1, \prec\rangle)^{2n+2}$. We define D, γ_0, $a(i)$, a, W, etc, as in 2.4.

We now continue to follow the proof of 2.4. Only case 2.2 requires special attention, as case 1, and case 2.1 are done exactly as in 2.4. Therefore we skip to case 2.2:

Suppose $E(a(2n)) \neq E(a(2n+1))$. Let $E(a(2n)) = E_v$, and $E(a(2n+1)) = E_w$. Suppose $E(a(2n)) \neq E(a(2n+1))$, but E_v and E_w are in the same component of the graph G_f. Let $v = v_0, v_1, \cdots, v_r = w$ be such that $E_{v_0}, \cdots, E_{v_r}$ is the unique path in G_f connecting E_v and E_w. For every $i < r$, let $\langle a(\eta_i), a(\mu_{i+1})\rangle$ denote the edge in G_f which connects $E(v_i)$ with $E(v_{i+1})$.

We define φ_s for $s \leq k$ inductively. φ_k is defined as in 2.1. If, for any $i < r$,

$s \neq v_i$, then φ_s is defined from φ_{s+1} as in 2.1.

If $s = v_i$, for any $1 \leq i < r-1$, we define the following: Suppose φ_{s+1} has been defined, and we are ready to define φ_s. Recall that $\langle a(\eta_{i-1}), a(\mu_i)\rangle$ connects $E_{v_{i-1}}$ and E_{v_i}, and $\langle a(\eta_i), a(\mu_{i+1})\rangle$ connects E_{v_i} and $E_{v_{i+1}}$. We are interested in $a(\mu_i), a(\eta_i)$, although $\langle a(\mu_i), a(\eta_i)\rangle$ is not an edge in G_f. Note that it can be that no such $s = v_i$'s exist (when $r = 1$).

If $a(\mu_i) = a(\eta_i)$, we continue as usual, and define φ_s as before.

Suppose $a(\mu_i) \neq a(\eta_i)$. Then $\langle a(\mu_i), a(\eta_i)\rangle$ belongs to one of $A \times A$, $A \times B$, or $B \times B$. As $A \perp A^*$, $A \perp B^*$, and $B \perp B^*$, by claim 2.6 we can construct, on either of the cases, a set of intervals $\{U_t^l \mid l \in \{0,1\}, t \in R_{v_i}\}$ which is $\{\eta_i, \mu_i\}$-OP. We do that for every $s = v_i$.

For $s = v$, suppose that φ_{v+1} has been defined. Recall that $a(2n), a(\eta_0) \in E_w$. As both $A \perp G^*$, and $B \perp G^*$, it is not important whether $a(\eta_0)$ is a member of A or of B, and we can construct, by claim 2.6, a set of intervals $\{U_t^l \mid l \in \{0,1\}$, $t \in R_v\}$ which is $\{2n, \eta_0\}$-OP.

For $s = w$, suppose that φ_{w+1} has been defined. Recall that $a(2n+1), a(\mu_r) \in E_w$, and, again, as $A \perp H$, and $B \perp H$, It is not important whether $a(\mu_{r-1})$ is a member of A, or of B, and we can construct a set of intervals $\{U_t^l \mid l \in \{0,1\}$, $t \in R_w\}$ which is $\{2n+1, \mu_r\}$-OR.

After we are done with the definitions of φ_s for $s < k$ we can choose, as in Theorem 2.1, $l(s)$ for $s < k$. Now let β and γ be such that for every $s < k$

$$a(\beta) \restriction R_s \in \prod_{i \in R_s} U_i^{l(s)} \text{ and } a(\gamma) \restriction R_s \in \prod_{i \in R_s} U_i^{1-l(s)}.$$

By the proof of 2.1, $f_\beta \cup f_\gamma \in P$. Note that $\{\langle a(\beta, 2n), a(\beta, 2n+1)\rangle$, $\langle a(\gamma, 2n), a(\gamma, 2n+1)\rangle\}$ is a composition of $\{\langle a(\beta, 2n), a(\beta, \eta_0)\rangle, \langle a(\gamma, 2n), a(\gamma, \eta_0)\rangle\}$

which is OP, of $\{\langle a(\beta,\eta_0), a(\beta,\mu_r)\rangle,\ \langle a(\gamma,\eta_0), a(\gamma,\mu_r)\rangle\}$ which is OP (as a composition of OP pairs), and of
$\{\langle a(\beta,\mu_r), a(\beta,2n+1)\rangle,\ \langle a(\gamma,\mu_r), a(\gamma,2n+1)\rangle\}$ which is OR. Hence there is an odd number of OR pairs which compose $\{\langle a(\beta,2n), a(\beta,2n+1)\rangle,$
$\langle a(\gamma,2n), a(\gamma,2n+1)\rangle\}$, so its type is OR. Hence, by theorem 2.8, it means that $\{\langle a_\beta, b_\beta\rangle, \langle a_\gamma, b_\gamma\rangle\}$ is OR, and we reach a contradiction. $\square$

3 The possible structure of K^H under the assumption of $MA_{\aleph_1}$

We will characterize the possible structure of $K^H/\cong$, when $K^H/\cong$ is finite, under the assumption of $MA_{\aleph_1}$. Lemma 1.11 (1) shows that $\preceq$ induces a partial ordering on $K^H/\cong$, hence we regard $\preceq$ as partial ordering on $K^H/\cong$. We will denote $K^H \cup \{\emptyset\}$ as K^{HZ}.

In this work will show that when $K^H/\cong$ is finite, then under the assumption of $MA_{\aleph_1}$, we can make $K^{HZ}/\cong$, with an involution function, isomorphic to any finite distributive lattice with involution. In this section we will state the main theorem, prove its easy side, and start proving the more complicated side, which will occupy us through the entire work.

3.1 Stating the main theorem and proving its easy direction

We start with several definitions.

Definition 3.1. *Let $\bar{g} : K \to K$ be the following function: for every $A \in K$, $\bar{g}(A) = \{-a \mid a \in A\}$. $\bar{g}$ will be denoted as $*$, and $\bar{g}(A)$ will be denoted as A^*.*

Note that for every $A, B \in K$, $A^* \in K$ and $(A^*)^* = A$, if $A \preceq B$ then $A^* \preceq B^*$, and if $A \cong B$, then $A^* \cong B^*$. Hence, on $\left\langle K^H/\cong, \preceq \right\rangle$, the operation $*$ can be considered as an automorphism of order 2. We will denote $*$ as the *involution automorphism*. $\bar{g}$ will be denoted as the *involution function*.

Definition 3.2. *Let $\langle L, \preceq\rangle$ be a finite lattice. We define $*$ as an automorphism of order 2 of $\langle L, \preceq\rangle$. Again, we denote $*$ as an* involution *on $\langle L, \preceq\rangle$.*

Note that for every $a, b \in L$ $(a^*)^* = a$, $(a \wedge b)^* = a^* \wedge b^*$, and $(a \vee b)^* = a^* \vee b^*$.

We now state our main theorem through the entire work:

Theorem 3.3. *Let $\langle L, \leq, *\rangle$ be a finite poset with an involution. Then the following are equivalent:*

(A) $\left(MA_{\aleph_1} + \left(\left\langle K^{HZ}/\cong, \leq, *\right\rangle \cong \langle L, \leq, *\rangle\right)\right)$ *is consistent with ZFC.*

(B) $\langle L, \leq, *\rangle$ *is a distributive lattice with an involution.*

We make an abuse of notation by denoting $K^H/\cong$ by K^H, and $K^{HZ}/\cong$ by K^{HZ}.

For example, in a universe which satisfies BA, $\left\langle K^{HZ}, \leq, *\right\rangle$ is isomorphic to a lattice with a single element. We want to find a way, in which $\left\langle K^{HZ}, \leq, *\right\rangle$ will be isomorphic to any given finite distributive lattice with involution.

We start with the easy direction of the theorem, that is $(A) \to (B)$. Given a finite poset with an involution $\langle L, \leq, *\rangle$, assume $MA_{\aleph_1}$, and suppose that $\left\langle K^{HZ}/\cong, \leq, *\right\rangle \cong \langle L, \leq, *\rangle$. We wish to see that $\langle L, \leq, *\rangle$ is a distributive lattice with an involution.

Definition 3.4. *Let $\mathcal{A} \subseteq K^H$; $\mathcal{A}$ generates K^H if every element of K^H is a shuffle of a countable subset of $\mathcal{A}$. K^H is countably generated if there is $\mathcal{A} \subseteq K^H$ such that $|\mathcal{A}| \leq \aleph_0$ and $\mathcal{A}$ generates K^H.*

The following is lemma 10.3 from [ARS], which basically states all we need for the proof.

Lemma 3.5. *Assume $(MA_{\aleph_1})$. Then:*

(a) K^H is a least σ-complete upper-semi-lattice, that is, every countable subset of K^H has a least upper bound.

(b) If K^H is countably generated, then K^{HZ} is a distributive complete lattice. We will denote the operations in K^{HZ} by $\curlywedge$ and $\curlyvee$.

(c) If $\langle K^H, \preceq \rangle$ is well founded, then for $A \in K$ there is a nowhere-dense subset B of A such that $A \backslash B$ is the ordered sum of members of K^H.

Proof. (a) Is just a reformulation of 1.11 (3). If $\{A_i \mid i < \omega\} \subseteq K^H$, Then let B be the shuffle of $\{A_i \mid i < \omega\}$. By 1.11 (3), $B \in K^H$ and is a least upper bound of $\{A_i \mid i < \omega\}$.

(b) Suppose K^H is countably generated. Then by (a), K^H is a complete upper-semi lattice. Let $\mathcal{A}$ be the set which generates K^H. Let $\{A_i \mid i < \omega\} \subseteq K^H$, and let $\mathcal{D} \stackrel{def}{=} \{A \in \mathcal{A} \mid (\forall i < \omega) A \preceq A_i\} \cup \{\emptyset\}$; then $\mathcal{D}$ is countable, and if C is the least upper bound of $\mathcal{D}$, then for every $A \in \mathcal{D}$, and $i \in \omega$, $A \preceq A_i$; hence, by 1.11 (3), for every $i < \omega$, $C \preceq A_i$. Suppose there is $X \in K^H$ such that for every $i < \omega$, $X \preceq A_i$, as X is a shuffle of some $\mathcal{B} \subseteq \mathcal{A}$; for every $B \in \mathcal{B}$, and $i < \omega$, $B \preceq A_i$. Hence $B \in \mathcal{D}$, so $B \preceq C$; then again, by 1.11 (3), $X \preceq C$, hence C is the greatest lower bound of $\{A_i \mid i < \omega\}$. Then $K^H \cup \{\emptyset\}$ is a complete lattice. As mentioned before, we will denote the operations in K^{HZ} by $\curlywedge$ and $\curlyvee$.

In order to show that K^{HZ} is distributive it is suffice to show one of the distributive laws. We show that for every $b_1, b_2, a \in K^H$, $(b_1 \curlyvee b_2) \curlywedge a = (b_1 \curlywedge a) \curlyvee (b_2 \curlywedge a)$. In fact we can show somewhat more: for every $a, \{b_i \mid i < \omega\}$,$(\bigvee_{i\in\omega} b_i) \bigwedge a = \bigvee_{i\in\omega}(b_i \bigwedge a)$. We do not know however whether the dual identity in this case holds.

Let $A \in K^{HZ}$ and for every $i \in \omega$, let $B_i \in K^{HZ}$. We want to show that $\bigvee_{i\in\omega}(A \curlywedge B_i) \cong A \curlywedge (\bigvee_{i\in\omega} B_i)$.

The inequality $\bigvee_{i\in\omega}(A \curlywedge B_i) \preceq A \curlywedge \bigvee_{i\in\omega} B_i$ holds in every lattice, thus we want to show that $A \curlywedge \bigvee_{i\in\omega} B_i \preceq \bigvee_{i\in\omega}(A \curlywedge B_i)$. To do that we use the following claim:

Claim 3.6. *Assume $MA_{\aleph_1}$. Let $A \in K^{HZ}$, and for every $i \in \omega$, let $B_i \in K^{HZ}$. Suppose that $A \preceq \bigvee_{i\in\omega} B_i$. Then for every $i < \omega$ there is $A_i \in K^{HZ}$ such that $A_i \preceq B_i$ and $A = \bigvee_{i\in\omega} A_i$.*

Proof. We can assume that each B_i is dense in $\Re$. Let $B = \bigcup_{i\in\omega} B_i$; then $B \in K$, and $B = \bigvee_{i\in\omega} B_i$. As $A \preceq B$, let $f : A \to B$ be an 1-1, OP function. Recall that C^m denotes a mixing of C. If $|C| \leq \aleph_0$, let $C^m = \emptyset$. For every $i \in \omega$, let $C_i = f^{-1}(B_i)$, so $C_i \preceq A$, and let $A_i = C_i^m$ (note that, C_i^m is a member of K^H regardless to whether C_i is a member of K or just an $\aleph_1$ size set; if the later occurs then $\preceq$ means simple embedding in the meaning of sets, C_i^m will still be defined in the usual way). By picking appropriate copies of C_i^m as A_i, one can assume that $A \subseteq \bigcup_{i\in\omega} A_i$, and since $\bigcup_{i\in\omega} A_i = \bigvee_{i\in\omega} A_i$ it follows that $A \preceq \bigvee_{i\in\omega} A_i$. On the other hand, As $C_i \preceq A$, and $C_i \preceq B_i$, then by 1.11 (4), for every $i < \omega$, $A_i \preceq A, B_i$. Hence $\bigvee_{i\in\omega} A_i \preceq A$, and $A_i \preceq B_i$. Hence $A = \bigvee_{i\in\omega} A_i$, and for every $i < \omega$, $A_i \preceq B_i$. $\square$

As $A \curlywedge \bigvee_{i\in\omega} B_i \preceq \bigvee_{i\in\omega} B_i$, then using the above claim on $A \curlywedge \bigvee_{i\in\omega} B_i$, there are $A_i \preceq B_i$ such that $\bigvee_{i\in\omega} A_i = A \curlywedge \bigvee_{i\in\omega} B_i$. Then $A_i \preceq A, B_i$, so $A_i \preceq A \curlywedge B_i$. Thus $\bigvee_{i\in\omega} A_i \preceq \bigvee_{i\in\omega}(A \curlywedge B_i)$, and as $\bigvee_{i\in\omega} A_i = A \curlywedge \bigvee_{i\in\omega} B_i$, it follows that $A \curlywedge \bigvee_{i\in\omega} B_i \preceq \bigvee_{i\in\omega}(A \curlywedge B_i)$, as required.

(c) Suppose that $\langle K^H, \preceq \rangle$ is well founded, and let $A \in K$. It suffices to show that every non-empty open interval of A contains a homogeneous subinterval. If A_1, A_2 are intervals of A and $A_1 \subseteq A_2$, then $A_1^m \subseteq A_2^m$. Recall that for every $C \in K$, C^m is a member of K^H. Let A_1 be an interval of A. Since K^H is well founded, A_1 has a non empty subinterval A_2 such that for every subinterval A_3 of A_2, $A_3^m \cong A_2^m$. We show that $A_2^m \preceq A_2$. Let $I \in \phi^{A_2}$. As $I \preceq I^m$, and $I^m \cong A_2^m$, by the definition of *mixing*, for every rational interval J, there is $I_J \subseteq I$, where $I_J \cong I$, and $\bigcup\{I_J \mid J$ is a rational interval$\} = A_2^m$; hence there is a family $\{g_i^I \mid i \in \omega\}$ of OP functions such that $Rng(g_i^I) = I$ and $\bigcup_{i\in\omega} Dom(g_i^I) = A_2^m$.

Then A_2^m, A_2 and $\{g_i^I \mid I \in \phi, i \in \omega\}$ satisfy the conditions of 1.3 (1), hence $A_2^m \preceq A_2$. $A_2 \preceq A_2^m$, hence by 1.11 (4), $A_2 \cong A_2^m$. $\square$

Now, as L is finite, then K^H is countably generated, and by the above lemma K^{HZ} is a distributive complete lattice, and as $\left\langle K^{HZ}/\cong, \leq, *\right\rangle \cong \langle L, \leq, *\rangle$, then $\langle L, \leq, *\rangle$ is a distributive lattice with an involution. That sums the proof of the easy side of theorem 3.3. $\square$

3.2 Beginning the proof of the complicated direction of the main theorem

For the rest of the work we will focus on the more complicated direction of the theorem, $(B) \rightarrow (A)$. That is, given a finite poset $\langle L, \preceq, *\rangle$, such that $\langle L, \preceq, *\rangle$ is a distributive lattice with an involution, , we want to construct a model in which it is consistent, along with $MA_{\aleph_1}$, that $\left\langle K^{HZ}/\cong, \leq, *\right\rangle \cong \langle L, \leq, *\rangle$.

Definition 3.7. *Let $\langle L, \preceq, *\rangle$ be a finite distributive lattice with an involution. $a \in L$ is* indecomposable *if for no $b, c < a$, $b \vee c = a$. Let $I(L)$ denote the set of indecomposables of L.*

Clearly $I(L)$ is closed under $*$, and every element of L is a sum of elements in $I(L)$. We also know that $I(L)$ determines L uniquely. Using $I(L)$, we will manage to construct a universe where $K^{HZ} \cong L$.

Proposition 3.8. *(a) Let $\langle A, \preceq, *\rangle$ be a finite partially ordered set with an involution. Then there is a unique distributive lattice with an involution L such that $\langle I(L), \preceq, *\rangle \cong \langle A, \preceq, *\rangle$.*

(b) Assume $MA_{\aleph_1}$. Let $\{A_i \mid i < n\} \subseteq K^H$ be such that: for no $j < n$, A_j is a shuffle of other members of $\{A_i \mid i < n\}$, and for every $A \in K^H$, A is a shuffle of some members of $\{A_i \mid i < n\}$. Then K^{HZ} is finite and $I(K^{HZ}) = \{A_i \mid i < n\}$.

Proof. (a) Let L be the collection of all downward closed (by $\preceq$) subsets of A. For $X, Y \in L$, let $X \preceq_L Y$ mean $X \subseteq Y$. Then L is a finite distributive lattice where $X \curlyvee Y = X \cup Y$, and $X \curlywedge Y = X \cap Y$, For $X \in L$, let $X^* = \{a^* \mid a \in X\}$. It is easy to see that the operation $*$ is an involution over L. Let $f : A \to L$ be $f(a) = \{x \mid x \preceq a\}$. Then $f(A)$ is actually $I(L)$, and f is an isomorphism between $\langle I(L), \preceq, *\rangle$, and $\langle A, \preceq, *\rangle$. L is unique, as $I(L)$ determines L uniquely.

(b) Assuming $MA_{\aleph_1}$, then by 3.5 (b), K^{HZ} is a distributive lattice. K^{HZ} is generated from $\{A_i \,|\, i < n\}$, therefore it is finite, and $I(K^{HZ}) = \{A_i \,|\, i < n\}$. $\square$

By the above proposition it is clear what has to be done in order to construct a universe in which $K^{HZ} \cong L$. We start with a universe V satisfying CH, and with a family $\{A_i \mid i \in I(L)\} \subseteq K^H$ such that no A_i is a shuffle of other A_i's and such that $i \mapsto A_i$ is an isomorphism between $\langle I(L), \preceq, *\rangle$ and $\langle \{A_i \,|\, i \in I(L)\}, \preceq, *\rangle$. We then construct a universe W, such that $V \subseteq W$, which satisfies MA, and in which every element of K^H is a shuffle of some members of $\{A_a \mid a \in I(L)\}$, and no A_a is a shuffle of other A_a's. Then K^{HZ} is a finite distributive lattice, and as $I(K^{HZ}) = \{A_a \mid a \in I(L)\}$, which is isomorphic to $I(L)$, then because of uniqueness, in such a universe $K^{HZ} \cong L$.

3.3 Creating a structure in K^H which is isomorphic to $I(L)$

For the rest of this work L is a fixed finite distributive lattice with an involution. We can assume that $I(L) = \{0, \cdots, n-1\}$, and we denote the partial ordering on $I(L)$ by $\preceq$. Recall that $*$ is an involution of $\langle\{0, \cdots, n-1\}, \preceq\rangle$.

Recall that $\bar{g}$ denote the function such that for every z, $\bar{g}(z) = -z$. For every function f, let $f^* = \bar{g} \circ f \circ \bar{g}^{-1}$. If F is a set of functions, let $F^* = \{f^* \mid f \in F\}$, and let $F^{-1} = \{f^{-1} \mid f \in F\}$. Note that, if f is OP, then so are f^*, and f^{-1}.

Definition 3.9. *Let $f \subseteq \Re * \Re$. f is* maximal OP function *if f is OP, and there is no OP function g such that $f \subsetneq g \subseteq \Re \times \Re$.*

We will now construct a family of sets in K^H, $\{A_i | i < n\}$, such that $\langle I(L), \preceq, *\rangle$ and $\langle\{A_a \mid a \in I(L)\}, \preceq, *\rangle$ are isomorphic. The following is lemma 10.5 from [ARS], and is due to Sierpinski [S].

Lemma 3.10. *(CH) There are $\{A_i \mid i < n\} \subseteq K^H$, all dense in $\Re$, such that:*

1. *If $i \preceq j$ then $A_i \subseteq A_j$.*
2. *If $i = j^*$ then $A_i = A_j^*$.*
3. *Define $B_i = A_i \setminus \bigcup_{j \preceq i} A_j$. Then $B_i \in K$, dense in $\Re$, and $i \not\preceq j \rightarrow B_i \perp A_j$.*

It is immediate from the lemma that for every $i < n$, $B_i^* = B_{i^*}$. We will sometimes refer to A_i as A_{τ_i} where $\tau_i = \{j \mid j \preceq i\}$. Note that for every $i, j < n$, $\tau_i \subseteq \tau_j$ if and only if $i \preceq j$.

Proof. Let $\{f_\alpha \mid \alpha < \aleph_1\}$ be an enumeration of all maximal OP functions. We define, by induction on $\alpha < \aleph_1$, a family of pairwise disjoint countable dense subsets of $\Re$, $\{B(i, \alpha) \,|\, i < n\}$, and countable families of 1-1 OP functions $\{F(i, \alpha) \,|\, i < n\}$. For every $\alpha < \aleph_1$, we denote $A(i, \alpha) = \bigcup_{j \preceq i} B(j, \alpha)$.

Our induction hypothesis is:

1. If $i = j^*$, then $B(i,\alpha) = B^*(j,\alpha)$.

2. For every $i < n$: If $f \in F(i,\alpha)$ then $f : \Re \Rightarrow \Re$ and $f(A(i,\alpha)) = A(i,\alpha)$; for every $x < y \in A(i,\alpha)$ there is $f \in F(i,\alpha)$ such that $f(x) = y$.

3. For every $i < n$, and $\beta < \alpha$, $F(i,\beta) \subseteq F(i,\alpha)$, and $B(i,\beta) \subseteq B(i,\alpha)$ (thus $A(i,\beta) \subseteq A(i,\alpha)$).

Let $U \subseteq n$ be such that for every $i < n$ $|U \cap \{i, i*\}| = 1$. For every $i \in U$, we define $B(i,0)$ as follows:

From every rational interval I we choose an element $a \in \Re$ such that neither a, nor $-a$ has been picked before by $B(i,0)$, or by other $B(j,0)$'s that were already defined. After defining $B(i,0)$ for every $i \in U$, we again define for every $i \in U$, $B(i^*,0) = B^*(i,0)$. Then every $B(i,0)$ is countable and dense in $\Re$, and all the $B(i,0)$'s are distinct.

For every $i \in U$, we define $F(i,0)$ as follows. For every $a < b, c < d$ in $A(i,0)$, there is an OP function $f'_{<a,b>}$ such that $f'_{<a,b>}(a) = b$, and $f'_{<a,b>}(A(i,0)) = A(i,0)$. ($f'_{<a,b>}$ is simply the isomorphism between two countable dense sets in $\Re$). Let $f_{<a,b>}$ be a maximal OP function such that $f'_{<a,b>} \subseteq f_{<a,b>}$. Let $F(i,0) = \{f_{<a,b>} \mid a < b \text{ in } A(i,0)\}$. We set $F(i^*,0) = F^*(i,0)$. Then $F(i,0)$ is as required, and as $A(i^*,0) = A^*(i,0)$, $F(i^*,0)$ meet the demands as well.

If δ is a limit ordinal, let $B(i,\delta) = \bigcup_{\alpha<\delta} B(i,\alpha)$, and $F(i,\delta) = \bigcup_{\alpha<\delta} F(i,\alpha)$.

Suppose $\{B(i,\alpha) \mid i < n\}$, and $\{F(i,\alpha) \mid i < n\}$ have been defined. We wish to define $\{B(i,\alpha+1) \mid i < n\}$. Let $\bar{B}(i,\alpha) = B(i,\alpha) \cup \{f_\beta(B(i,\alpha)) \mid \beta < \alpha\}$.

Again, let $U \subseteq n$ be such that for every $i \in n$, $|U \cap \{i, i^*\}| = 1$. For $x \in \Re$ and a set of functions F, let $cl(x,F)$ denote the closure of x under F and F^{-1}. As every $F(i,\alpha)$, and every $B(j,\alpha)$ (for $j < n$) are countable, and as for any $a < b$

in $A(i,\alpha)$, there is $f \in F(i,\alpha)$ such that $f(a) = b$, it is easy to construct a set $\{x_i \mid i \in U\}$ such that:

1) For every $i \in U$, $cl(x_i, F(i,\alpha)) \cap cl(-x_i, F^*(i,\alpha)) = \emptyset$.

2) $cl(x_i, F(i,\alpha))$ is dense in $\Re$.

3) For every $i \in U$, $cl(x_i, F(i,\alpha)) \cap \bigcup_{j<n} \bar{B}(j,\alpha) = \emptyset$.

Now for every $i < n$: If $i \in U$ and $i \neq i^*$, let $B(i,\alpha+1) = B(i,\alpha) \cup cl(x_i, F(i,\alpha))$; then for i^*, let $B(i^*,\alpha+1) = B(i^*,\alpha) \cup cl(-x_i, F^*(i,\alpha))$. If $i = i^*$, let $B(i,\alpha+1) = B(i,\alpha) \cup cl(x_i, F(i,\alpha)) \cup cl(-x_i, F^*(i,\alpha))$. Since for every i, $F^*(i,\alpha) = F(i^*,\alpha)$, it follows that $B^*(i,\alpha+1) = B(i^*,\alpha+1)$. By the choice of the x_i's, whenever $i \neq j$, $B(i,\alpha+1) \cap B(j,\alpha+1) = \emptyset$.

As in the basic case, for every $a < b$ in $A(i,\alpha+1)$, we take a maximal OP function f such that $f(a) = b$, and $f_{<a,b>}(A((i,\alpha+1)) = A((i,\alpha+1))$. We denote f as $f_{<a,b>}$, denote $f'(i,\alpha+1) = \{f_{<a,b>} \mid a < b \text{ in } A((i,\alpha+1))\}$, and denote $f(i,\alpha+1) = f'(i,\alpha+1) \cup f(i,\alpha)$. As for every $f \in F(i,\alpha)$, and $x \in cl(x_i, F(i,\alpha))$, $f(x) \in B(i,\alpha+1)$, then $f(A(i,\alpha+1)) = A(i,\alpha+1)$; hence $f(i,\alpha+1)$ is defined well, plus the $B(i,\alpha+1)$'s and the $F(i,\alpha+1)$'s meet the induction requirement.

For every $i < n$, let $A_i = \bigcup_{\alpha<\aleph_1} A(i,\alpha)$. As for every $\alpha < \aleph_1$, $A(i,\alpha)$ is a countable dense set, and in the $\alpha+1$ step, a distinct countable dense set was added to $A(i,\alpha)$ to form $A(i,\alpha+1)$ then A_i is a member of K. As for every $a, b \in A_i$, there is $\alpha < \aleph_1$ such that $a, b \in A(i,\alpha)$, there is $f \in F(i,\alpha)$ such that $f(a) = b$; by the construction of A_i ,$f(A_i) = A_i$ and f is an automorphism of $\langle A_i, < \rangle$, hence $A \in K^H$. It is immediate to see, by construction of the A_i's, that if $i \preceq j$, then $A_i \subseteq A_j$, and that $A_{i^*} = A_i^*$.

Now, suppose by contradiction that $i \not\preceq j$ but B_i, A_j are not far. As $A_j = \bigcup_{k \preceq j} B_j$, then for some $k \neq i$, B_i, B_k are not far. Then there is a maximal OP function f such that $|f \cap B_i \times B_k| = \aleph_1$. Then in our enumeration, for some

$\alpha < \aleph_1$, $f = f_\alpha$. But by the construction, for every $\beta > \alpha$,

$f_\alpha(B(i,\beta)) \cap (B(k,\beta) \setminus \bigcup\{B(k,\gamma) \mid \gamma < \beta\}) = \emptyset$, hence $|f_\alpha(B(i) \cap B(k))| \leq \aleph_0$, a contradiction. $\square$

3.4 Defining Farness Formulas

Now that we found a structure $\{A_i \mid i < n\}$ such that $\langle \{A_i \mid i < n\}, \preceq, * \rangle \cong \langle I(L), \preceq, * \rangle$, we will take one element of K^H at a time, and make it isomorphic to a shuffle of some A_i's, by using BA-forcing (see section 2). By doing that, we would like to maintain the structure of the A_i's above; which means that if for $i, j < n$, $i \not\preceq j$, then after the forcing we use, it will still be that $A_i \not\preceq A_j$.

For $C_0 \cdots C_{k-1}$ in K, we define $\bigwedge_{i<k} C_i = 0$ if there is no such $C \in K$ such that for each $i < k$, $C \preceq C_i$. If there is such $C \in K$ such that $(\forall i < k)(C \preceq C_i)$, we will write $\bigwedge_{i<k} C_i \neq 0$. Notice that as K is not a lattice, there are no meets in K, so $\bigwedge_{i<k} C_i$ has no meaning.

Let $\{x_i \mid i < n\}$ and $\{y_\tau \mid \tau \subseteq n\}$ be variables. We will later use formulas containing these variables, where B_i is assigned for x_i, and A_i for y_{τ_i}.

Let $\epsilon \in \{0, 1\}$ be such that if z is a variable, then for $\epsilon = 1$, z^ϵ denotes z^*, and for $\epsilon = 0$, z^ϵ denotes z. We will define a *Farness-Formula* (or F-Formula) as a formula of the form $\bigwedge_{i \in I} z_i^{\epsilon_i} = 0$, where $\epsilon_i \in \{0, 1\}$, and $\{z_i \mid i \in I\}$ is any set of variables from $\{x_i \mid i < n\} \bigcup \{y_\tau \mid \tau \subseteq n\}$. Note that, any F-formula can be considered of the form $\bigwedge_{i \in I} x_i^{\epsilon_i} \wedge \bigwedge_{\tau \in J} y_\tau^{\epsilon_\tau} = 0$ where $I \subseteq n$ and $J \subseteq P(n)$.

In K, let s be an assignment on $\{x_i \mid i < n\} \bigcup \{y_\tau \mid \tau \subseteq n\}$ such that $s(x_i) = B_i$, and $s(y_{\tau_i}) = A_i$. Suppose, for example, that $\chi_0 \equiv (x_i \wedge x_j = 0)$ for some $i \neq j$. As

$B_i \perp B_j$, it follows that $K \models \chi_0[s]$. Same, suppose $\chi_1 = (x_i \wedge y_{\tau_j} = 0)$ for some i, j, where $i \notin \tau_j$. Then $K \models \chi_1[s]$, as $B_i \perp A_{\tau_j}$. Thus, the collection of $\{\chi \mid \chi$ is an F-formula of the sort of χ_0, or χ_1, and $K \models \chi[s]\}$ provides us with the idea of what the structure of $\langle \{A_i \mid i < n\}, \preceq, *\rangle$ looks like. In this section we will define this collection more accurately.

Let $\langle L', \preceq, *\rangle$ be a finite distributive lattice with an involution $*$, and let 0 be the minimum of L'. We call $a \in L'$ an *atom* of L' if there is no $b \in L'$, $b \neq 0$ such that $b \preceq a$. Let $A(L')$ denote the set of atoms of L'. Suppose $A(L') = \langle a_0, \cdots, a_{n-1}\rangle$. It is easy to see that $A(L')$ is closed under involution.

For variables $\{x_i \mid i < n\}$, and $\{y_\tau \mid \tau \subseteq n\}$, an assignment s' on L' will be called a *fine assignment* if for every $i \in n$, $(s'(x_i)) = a_i$, $(s'(x_i))^* = s'(x_{i^*})$, and for every $\tau \subseteq n$, $s'(y_\tau) = \bigvee\{s'(x_i) \mid i \in \tau\}$. Note that, $s'(y_{\tau^*}) = s'(y_\tau^*)$, where $\tau^* = \{j^* \mid j \in \tau\}$.

we want every variable y_τ to have this following property, which will be denoted as the *maximal property* of y_τ: there is no $\tau_1 \subseteq n$, such that $\tau \subseteq \tau_1$, and if $s'(y_{\tau_1}) = \bigvee\{s'(x_i) \mid i \in \tau_1\}$, then $s'(y_{\tau_1}) = s'(y_\tau)$.

Note that this property is very easy to reach by removing variables that do not satisfy this property.

For the rest of the section, we assume that all the y_τ's have the maximal property.

Definition 3.11. *As before, let $\{x_i \mid i < n\}$ and $\{y_\tau \mid \tau \subseteq n\}$ be variables, let $\varphi_0(x_0, \cdots, x_{n-1})$ be the following formula:*

$$\varphi_0 \equiv \left(\bigwedge_{i \neq j} (x_i \wedge x_j = 0) \right) \wedge \left(\bigwedge_{i \neq j^*} (x_i \wedge (x_j)^* = 0) \right)$$

For an F-formula χ we will define $\varphi_0 \Rightarrow \chi$ if for any finite distributive lattice

$\langle L', \preceq, *\rangle$, with an involution and a minimum 0, such that $A(L') = \{0, \cdots, n-1\}$, and for any fine assignment s' on L'; if $L' \models \varphi_0[s']$, then $L' \models \chi[s']$.

For the rest of the section, let L' denote such a finite distributive lattice, and let s' be a fine assignment over L'.

Let χ_0 be an F-formula such that $\chi_0 \equiv (\bigwedge_{i \in I} z_i^{\epsilon_i} = 0)$ and z^ϵ be a variable as defined before; then the F-formula $\chi \equiv (\bigwedge_{i \in I} z_i^{\epsilon_i} \wedge z^\epsilon = 0)$ is called an *elaboration* of χ_0. In the same way, if $\chi_0 \equiv (\bigwedge_{i \in I} z_i^{\epsilon_i} = 0)$, $\chi_1 \equiv (\bigwedge_{i \in J} (z_i')^{\epsilon_i'} = 0)$ are F-formulas, Then the F-formula $\chi \equiv (\bigwedge_{i \in I} (z_i)^{\epsilon_i} \wedge \bigwedge_{i \in J} (z_i')^{\epsilon_i'} = 0)$ is called an elaboration of χ_0 (or of χ_1). For the sake of convenience we will also say that χ_0 is an elaboration of itself.

Claim 3.12. *Let χ_0 be an F-formula, and Let χ be an elaboration of χ_0. Then if $\varphi_0 \Rightarrow \chi_0$, then $\varphi_0 \Rightarrow \chi$.*

Proof. Obvious.

Definition 3.13. *Let $\varphi_1 \equiv \{\bigwedge \chi \mid \chi$ is an F-formula, and $\varphi_0 \Rightarrow \chi\}$.*

It is φ_1 that we will be interested with as a collection of farness formulas that, when referring to K, will describe the structure $\langle \{A_i \mid i < n\}, \preceq, *\rangle$.

Note that sometimes, in some formulas, we will use only parts of the variables x_i's and y_τ's; then if s is an assignment such that $Dom(s) \subseteq \{x_i \mid i < n\} \cup \{y_\tau \mid \tau \subseteq n\}$, we say that $\varphi_1[s]$ holds ($\models \varphi_1[s]$), if $\chi[s]$ holds for every conjunct χ of φ_1 whose variables belong to $Dom(s)$.

Lemma 3.14. 1. *Every conjunct of φ_0 is a conjunct of φ_1.*

2. *Let $\chi \equiv (x_i \wedge y_\tau = 0)$. Then: χ is a conjunct of φ_1 if and only if $i \notin \tau$.*

3. *Let $\chi \equiv (y_{\tau_0} \wedge y_{\tau_1} = 0)$. Then χ is a conjunct of φ_1 if and only if $\tau_0 \cap \tau_1 = \emptyset$.*

4. *If χ is a conjunct of φ_1, then χ is an elaboration of F-formula of the type from cases (1),(2), or (3).*

5. *Let $\chi \equiv (\bigwedge_{i\in I} z_i^{\epsilon_i} = 0)$ be a conjunct of φ_1. Then $\chi^* \equiv (\bigwedge_{i\in I} z_i^{1-\epsilon_i} = 0)$ is a conjunct of φ_1 as well.*

Proof.

1. Obvious, as every conjunct χ of φ_0 is an F-formula, and $\varphi_0 \Rightarrow \chi$.

2. Suppose $i \notin \tau$, then for every $j \in \tau$ $i \neq j$, hence $x_i \wedge x_j = 0$ is a conjunct of φ_0; now suppose there was $c \in L$, $c \neq 0$, such that $c \preceq s'(x_i)$, and $c \preceq s'(y_\tau)$. As $s'(x_i)$ is an atom, it follows that $c = s'(x_i)$, and because of the maximal property of y_τ, $i \in \tau$, a contradiction.

   On the other hand, suppose $i \in \tau$, then in L', $s'(x_i) \wedge \bigvee\{s'(x_j) \mid j \in \tau\} = s'(x_i)$, hence χ is not a conjunct of φ_1.

3. Is proved in the same way as (2).

4. Obvious. As every F-formula contains at least two variables from $\{x_i \mid i < n\} \bigcup \{y_\tau \mid \tau \subseteq n\}$.

5. Obvious.

Now that we have a good idea of the nature of φ_1, we can take the discussion back to K.

Lemma 3.15. *Let $C_0 \cdots C_{k-1} \in K$, and let s_0 be an assignment such that $s_0(x_i) = C_i$, and $s_0(y_\tau) = \bigcup_{i\in\tau} C_i$. Suppose that $K \models \varphi_0[s_0]$. then $K \models \varphi_1[s_0]$.*

Proof. Knowing the types of Farness-Formulas in φ_1 from lemma 3.14, the proof is easy.

Lemma 3.16. *Let $C_0, \cdots, C_{k-1}, D_0, \cdots, D_{k-1} \in K$, and let s_0 be an assignment such that $s_0(x_i) = C_i$, and $s_0(y_\tau) = D_i$. Let $C'_0, \cdots, C'_{k-1}, D'_0, \cdots, D'_{k-1}$ be in K as well, such that for every $i < k$, $C'_i \subseteq C_i$, $D'_i \subseteq D_i$, and let s_1 be an assignment such that $s_1(x_i) = C'_i$, and $s_1(y_\tau) = D'_i$.*

Then if $K \models \varphi_1[s_0]$, then $K \models \varphi_1[s_1]$.

Proof. Let $\chi \equiv (\bigwedge_{i \in I} z_i^{\epsilon_i} = 0)$ be a conjunct of φ_1, and suppose that there was $D \in K$ such that for every $i \in I$, $D \preceq s_1(z_i^{\epsilon_i})$. As for every $i \in I$, $s_1(z_i^{\epsilon_i}) \subseteq s_0(z_i^{\epsilon_i})$, it follows that for every $i \in I$, $D \preceq s_0(z_i^{\epsilon_i})$; hence $K \models \neg\varphi_1[s_0]$, a contradiction.

Note that $K \models \chi[s]$ if and only if $K \models \chi^*[s]$. As if there was $C \in K$ such that $C \preceq s(z_i^{1-\epsilon_i})$ for every $i \in I$, then $C^* \preceq s(z_i^{\epsilon_i})$ for every $i \in I$.

Lemma 3.17. *$K \models \varphi_0[B_0, \cdots, B_{n-1}]$.*

Proof. The proof is immediate by the construction of the A_i's and the B_i's in lemma 3.10.

Corollary 3.18. *Let s be the assignment such that $s(x_i) = B_i$, and $s(y_{\tau_i}) = A_i$. Then $K \models \varphi_1[B_0, \cdots, B_{n-1}, A_0, \cdots, A_{n-1}]$.*

Proof. Followed straight from 3.15, and 3.17, as for every $i \in n$, $A_i = \bigcup_{j \preceq i} B_j$.

For $B \in K$, a residue of B will be a certain $B' \in K$, such that $B' \subseteq B$. As the process of making every member of K^H isomorphic to some shuffle of the A_i's continues, it can be that some of the B_i's that were far from each other, will become near. What we want to show is that it is enough that K, with the assignment of the residues of the B_i's, and the A_i's, will suffice φ_1, in order to keep $\langle \{A_i \mid i < n\}, \preceq, * \rangle$ isomorphic to $\langle I(L), \preceq, * \rangle$.

Theorem 3.19. *Let W be a universe, such that $V \subseteq W$, and V contains all the A_i's and the B_i's etc. Suppose that in V $\langle \{A_i \mid i < n\}, \preceq, * \rangle \cong \langle I(L), \preceq, * \rangle$. Then* *if, in W, there are residues B_i' of B_i and an assignment s_0 such that $s_0(x_i) = B_i'$, $s_0(y_{\tau_i}) = A_i$, and $K \models \varphi_1[s_0]$, then, still in W, $\langle\{A_i \mid i < n\}, \preceq, *\rangle \cong \langle I(L), \preceq, *\rangle$.*

Proof. Suppose that, in W, $\langle\{A_i \mid i < n\}, \preceq, *\rangle \ncong \langle I(L), \preceq, *\rangle$. As in V, the two models were isomorphic, then, in W, there are $i, j < n$, such that $i \npreceq j$, but $A_i \preceq A_j$. As $i \npreceq j$, there is $k < n$ such that $k \in \tau_i$ but $k \notin \tau_j$. Therefore by lemma 3.14, $x_k \curlywedge y_{\tau_j} = 0$ is a conjunct of φ_1. But as $s_0(x_k) = B_k'$, and $B_k' \subseteq B_k \subseteq A_i \subseteq A_j$, It follows that $\neg(B_k' \curlywedge A_j = 0)$, so K does not suffice $\varphi_1[s_0]$, a contradiction. $\square$

So to keep $\langle\{A_i \mid i < n\}, \preceq, *\rangle \cong \langle I(L), \preceq, *\rangle$, in the process of making members of K^H isomorphic to shuffles of some of the A_i's, we will define an assignment s_0 on K, for which, for every y_{τ_i}, $s(y_{\tau_i}) = A_i$, but the $s(x_i)$'s will be special residues of the B_i's. We will see that K, with this assignment, suffices φ_1, and this will ensure us that the isomorphism between $\langle\{A_i \mid i < n\}, \preceq, *\rangle$ and $\langle I(L), \preceq, *\rangle$ is being kept.

From now on instead of using the formality of $K \models \varphi_1[\cdots]$ we will simply say that the structure remains solid (it is obvious we are referring the structure of $\langle\{A_i \mid i < n\}, \preceq, *\rangle$). Also, for the rest of this work, let s be the assignment which maps each x_i to B_i and each y_{τ_i} to A_i. Claim 3.18 shows that $K \models \varphi_1[s]$.

4 Outlining the details of the main theorem proof

In the previous section we defined $\{A_i \mid i < n\}$, $\{B_i \mid i < n\}$ and formed a formula φ_1, which maintaining it will insure us that the structure of $\langle\{A_i \mid i < n\}, \preceq, *\rangle$ is remained stable. In this section we will be more specific in the details of the proof of the complicated side of the main theorem (theorem 3.3). We will also explain the limit stage, as the forcing we define will be iterated forcing, and solve a certain complicity which might arise when combining this forcing with the iterated forcing which yields $MA_{\aleph_1}$. For solving this complicity we use the tool known as the *Explicit Contradiction Method.*

4.1 Defining the iterated forcing and explaining the limit stage

Let $\{a(i,\alpha) \mid \alpha < \aleph_1\}$ be a $1-1$ enumeration of A_i. For every $i < n$ and every rational interval I, let $\{b(\alpha, i, I) \mid \alpha < \aleph_1\}$ be a $1-1$ enumeration on $B_i \cap I$ such that if $b(\alpha, i, I) = a(i, \alpha')$, and $b(\beta, i, I) = a(i, \beta')$ then $\alpha < \beta \leftrightarrow \alpha' < \beta'$.

Let F_i be the following filter: $F_i = \{B \subseteq B_i \mid$ for every rational interval I, $\{\alpha \mid b(\alpha, i, I) \in B\}$ contains a club$\}$. Note that, for every $B \in F_i$, B is in K and dense in B_i. We can assume that if $i = j^*$ then when enumerating B_i, $b(\alpha, i, I) = -b(\alpha, j, I^*)$ for every $\alpha < \aleph_1$. Thus $\bar{g}$ becomes an isomorphism between F_i and F_j (in other words $F_{i^*} = F_i^*$ where $F_i^* = \{B^* \mid B \in F_i\}$).

We now outline in more detail the proof of the second half of the main theorem(theorem 3.3). We start with a universe V satisfying CH, and with $\{A_i \mid i < n\}$, $\{B_i \mid i < n\}$, $\{F_i \mid i < n\}$ as described above. We define by induction on $\nu < \aleph_2$ a finite support iteration of c.c.c forcing sets $\{P_\nu \mid \nu \leq \aleph_2\}$, $\{\pi_\nu \mid \nu < \aleph_2\}$, and a sequence $\{\langle \check{B}(\nu, 0), \cdots, \check{B}(\nu, n-1)\rangle \mid \nu < \aleph_2\}$ such that for every ν and i, $\check{B}(\nu, i)$ is a P_ν-name,

$$\Vdash_{P_\nu} (\forall i < n)(\check{B}(\nu, i) \in F_i) \wedge (\check{B}^*(\nu, i) = \check{B}(\nu, i^*)), \text{ and}$$

$$\Vdash_{P_\nu} (K \models \varphi_1[\check{B}(\nu, 0), \cdots, \check{B}(\nu, n-1), A_0, \cdots, A_{n-1}]).$$

where each $\check{B}(\nu, i)$ is replacing x_i, and each A_i is replacing y_{τ_i}.

We prepare in advance a list of tasks. There are two kinds of tasks: the first one is designed in order to take care that $\Vdash_{P_{\aleph_2}} MA_{\aleph_1}$, and the second is to assure that $\Vdash_{P_{\aleph_2}} \langle K^H \cup \{\emptyset\} / \cong, \preceq, * \rangle \cong \langle L, \leq, * \rangle$.

We start by explaining the limit stage.

If δ is a limit ordinal, we will define P_δ in the usual way. We want to check that the induction hypothesis holds. Naturally, there are two cases to consider.

Case 1: $cf(\delta) = \aleph_0$.

Let $\{\alpha_j \mid j < \omega\}$ be an increasing sequence converging to δ. Let G be P_δ generic. Then in $V[G]$, for every $i < n$, $\bigcap_{j\in\omega} B(\alpha_j, i) \in F_i$, and $(\bigcap_{j\in\omega} B(\alpha_j, i))^* = \bigcap_{j\in\omega} B(\alpha_j, i^*)$, hence there is a sequence of P_δ-names, $\{\check{B}(\delta, 0), \cdots, \check{B}(\delta, n-1)\}$, such that $\Vdash_{P_\delta} (\forall i < n)(\check{B}(\delta, i) \in F_i) \wedge (\check{B}^*(\delta, i) = \check{B}(\delta, i^*))$.

Suppose that $\chi = (\bigwedge_{j\in J} z_j^{\epsilon_j} = 0)$ is a conjunct of φ_1, and if G is P_δ generic, then in $W = V[G]$, $K \models \neg\chi[s]$, where s is the usual assignment which puts $B(\delta, i)$ is x_i, and A_i in y_{τ_i}; for convenience, for every $\alpha < \delta$ let s_α be the assignment which puts A_i in y_i, and $B(\alpha, i)$ in x_i. Let $H \in W$ be an uncountable set, such that $H \subseteq \prod_{j\in J} s(z_j^{\epsilon_j})$, where every two members of H are OP. Then there are H_k, $k \in \omega$ such that $H = \bigcup_{k\in\omega} H_k$, and $H_k \in V[G \cap P_{\alpha_k}]$. Some H_k is uncountable, and as in $V[G]$ for every $i < n$, $B(\delta, i) \subseteq B(\alpha_k, i)$, then $H_k \subseteq \prod_{j\in J} s_{\alpha_k}(z_j^{\epsilon_j})$, where s_{α_k} is the assignment which puts $B(\alpha_k, i)$ in x_i, and A_i in y_{τ_i}, a contradiction.

Case 2: $cf(\nu) = \aleph_1$

The following claim is lemma is 9.8 from [ARS]

Claim 4.1. *(CH)*

Let $\lambda < \aleph_1$; for every $i < \lambda$, let $C_i \subseteq \aleph_1$ be a club. Then there is a club $C \subseteq \aleph_1$ such that for every $i < \lambda$, C is almost a subset of C_i (except for a countable set) .

Proof. Let $M = \langle \lambda, <, R \rangle$ where $R = \{\langle \alpha, i \rangle \mid \alpha \in C_i\}$. Let C be the club of the elementary substructures of M. For every $\delta \in C$, let $N_\delta \preceq M$ be such that $|N_\delta| \cap \aleph_1 = \delta$. Then for every $i < \lambda$, from elementary substructures considerations, there is a set of elements of C_i which is unbounded in N_δ. As C_i is a club, $\delta \in C_i$. Hence, for every $i < \lambda$, there is a point from which $C \subseteq C_i$. $\square$

Let G be P_δ generic. Then in $V[G]$, if $U \subseteq n$ is such that for every $i < n$ $|U \cap \{i, i^*\}| = 1$, then for every $i \in U$, and for every rational interval I, there is a club from the above claim, $C_{i,I} \subseteq \aleph_1$ which is almost a subset of $C_{\alpha,i,I}$ for every $\alpha < \delta$, where $C_{\alpha,i,I}$ is the club which defines $B(\alpha, i)$ at the interval I. Then, in $V[G]$, for every $i \in U$, there is a subset of B_i, which is defined by $C_{i,I}$ at the interval I. Hence there is a sequence of P_δ-names, $\{\check{B}(\delta, 0), \cdots, \check{B}(\delta, n-1)\}$ as required.

We again assume that χ is a conjunct of φ_1 of the form $(\bigwedge_{j \in J} z_j^{\epsilon_j} = 0)$, and if G is P_δ generic, then in $W = V[G]$, $K \models \neg\chi[s]$, where s is as in previous case; suppose $|J| = k$, and for $j < k$ let $D_j = s(z_j^{\epsilon_j})$. Let $H \in W$ be an uncountable set, as in the previous case, where every two members of H are OP.

Let τ be a P_δ-name for H. We define $\tau = \{\tau_\alpha \mid \alpha < \aleph_1\}$, and $\{q_\alpha \mid \alpha < \aleph_1\}$ such that for every $\alpha < \aleph_1$, $q_\alpha \in P_\delta$, and $q_\alpha \Vdash \tau_\alpha = \langle d_\alpha(0), d_\alpha(k-1) \rangle$, where for every $j < k$, $d_\alpha(j) \in D_j$. We assume that the q_α's form a Δ-system, and its kernel is r where $r \subset \beta'$ for some $\beta' < \delta$. By a standard argument there is $p \in P_\delta$ such that $p \Vdash |\{\alpha \mid t_\alpha \in \check{G}\}| = \aleph_1$. Suppose that $supt(p) \subseteq \beta'$ as well, and so is $supt(p')$, where p' is the P_δ condition which says that τ is of the size $\aleph_1$ etc. As for every $i < n$, in $V[G]$, starting from some $\beta_i < \aleph_1$, $B(\delta, i) \subseteq B(\beta_i, i)$, then if $\beta = max\{\beta_i \mid i < n; \beta'\}$, then in $V[G_\beta]$ where $G_\beta = G \cap P_\beta$, $\tau^{G_\beta} \subseteq \prod_{i \in J} s_\beta(z_j^{\epsilon_j})$, where s_β is as in previous case; the as size of τ^{G_β} is $\aleph_1$, and every two of its members are OP, hence we reach a contradiction. That sums the limit case. $\square$

4.2 Handling the tasks of the MA forcing

We now turn to the successor stage of the induction. The next section will deal with the case where the ν task is a P_ν-name of a member A of K.

In this section we will assume that the ν's task is a P_ν-name π such that $\Vdash_{P_\nu}$ "π is c.c.c", i.e. a task in the list of tasks for the proof of $MA_{\aleph_1}$.

If $\Vdash_{P_\nu * \pi} (K \models \varphi_1[\check{B}(\nu, 0), \cdots, \check{B}(\nu, n-1), A_0, \cdots, A_{n-1}])$, then we define $P_{\nu+1}$ to be $P_\nu * \pi$ and $\check{B}(\nu+1, i)$ to be $\check{B}(\nu, i)$, and proceed, as there is no difficulty.

Otherwise there seems to be a problem: on one hand, to achieve $MA_{\aleph_1}$, forcing with $P_\nu * \pi$ seems necessary. On the other hand, forcing with $P_\nu * \pi$ will jeopardize our attempts to maintain the structure of the A_i's. Therefore we shall find a c.c.c forcing set R, such that in V^R, $P_\nu * \pi$ is no longer c.c.c, and more important, in V^R, $(K \models \varphi_1[\check{B}(\nu, 0), \cdots, \check{B}(\nu, n-1), A_0, \cdots, A_{n-1}])$. This method is denoted as the *Explicit Contradiction Method*, and it appears in [ARS] section 2.

To be more accurate, we define $P_{\nu+1}$ to be the forcing set R, which is constructed in the following lemma, which is lemma 10.6 from [ARS] (using $P_\nu * \pi$ for Q). The proof presented here is elaboration of the proof from [ARS]. Note that techniques from the Isomorphism forcing (see section 2) serve as a ground for this proof.

Lemma 4.2. *(CH) Let s be an assignment such that $K \models \varphi_1[s]$. Suppose Q is a c.c.c forcing set such that $\Vdash_Q (K \models \neg\varphi_1[s])$. Then there is a c.c.c forcing set R of power $\aleph_1$ such that $\Vdash_R (K \models \varphi_1[s])$, and $\Vdash_R$ (Q is not c.c.c).*

Proof. As $K \models \varphi_1[s]$, There is a conjunct of φ_1, $\chi \equiv (\bigwedge_{i\in I}(z_i)^{\epsilon_i} = 0)$, such that in V, $K \models (\chi[s])$, but Q forces that $K \models \neg(\chi[s])$. For every $i \in I$, let $C_i = s(z_i^{\epsilon_i})$, $|I| = k$, and Let $m \geq 2$. Let M be a model that encodes all the relevant information, and C_M be the club of elementary substructures of M.

A set of k-tuples, $\langle a_0, \cdots, a_{k-1}\rangle$,$\langle b_0, \cdots, b_{k-1}\rangle$, is called *OP* if for every $i < j < k$: $a_i < b_i$ if and only if $a_j < b_j$. Note that, if there was $D \in K$ such that $D \preceq C_i$ for every $i < k$, Then there was a $B \subseteq \prod\limits_{i<k} C_i$ such that if $b_\alpha = \left\langle b_\alpha^0, \cdots, b_\alpha^{k-1}\right\rangle$, and $b_\beta = \left\langle b_\beta^0, \cdots, b_\beta^{k-1}\right\rangle$ are members of B then $\langle b_\alpha, b_\beta\rangle$ is OP.

Let τ be a Q-name such that $\Vdash_Q (\tau \subseteq \prod_{i<k} C_i)\wedge(|\tau| = \aleph_1)\wedge$(every two element pair subset of τ is OP). For $m \geq 2$ let $\{\langle q_\alpha, a(\alpha,0), \cdots, a(\alpha, m-1)\rangle \mid \alpha < \aleph_1\}$ be such that:

1) For every $\alpha < \aleph_1$, $q_\alpha \in Q$, and for every $i < m$, $q_\alpha \Vdash_Q \check{a}(\alpha, i) \in \tau$.

2) For every $\alpha < \aleph_1$ and $i < j < m$, there is $\gamma \in C^M$ such that $a(\alpha, i) < \gamma \leq a(\alpha, j)$, and for every $\alpha < \beta < \aleph_1$, there is $\gamma \in C^M$ such that $a(\alpha, m-1) < \gamma \leq a(\beta, 0)$.

Note that every $a(\alpha, i)$ is a k-tuple, already exists in V, and that when saying $a(\alpha, i) < \gamma$, we mean that in the $\aleph_1$ enumeration of $\Re$, all of $a(\alpha, i)$ elements are smaller than γ.

If, for $i \neq j$, $q_\alpha \Vdash_Q (\check{a}(\alpha, i) \in \tau \wedge \check{a}(\alpha, j) \in \tau)$, then it is obvious that $\langle a(\alpha, i), a(\alpha, j)\rangle$ is OP. As for that, notice q_α, q_β are compatible if and only if for every $i, j < m$, $\langle a(\alpha, i), a(\beta, j)\rangle$ is OP.

We say that q_α and q_β are *explicitly contradictive* if for some $i < m$, $\langle a(\alpha, i), a(\beta, i)\rangle$

is not OP. Clearly, if q_α and q_β are explicitly contradictive, then they are incompatible in Q.

Let $R = \{\sigma \subseteq \aleph_1 \mid \sigma$ is finite, and for every distinct $\alpha, \beta \in \sigma$, q_α and q_β are explicitly contradictive$\}$. Let $\sigma_1 \leq \sigma_2$, if $\sigma_2 \subseteq \sigma_1$. Let $1_R = \emptyset$.

We will show that R is c.c.c, $\Vdash_R$ (Q is not c.c.c), and $\Vdash_R (K \models \varphi_1[s])$ (to be more accurate, we will find $r \in R$ such that the forcing set $R \restriction r$ satisfies these conditions).

1) $\Vdash_R (K \models \varphi_1[s])$.

Suppose by contradiction that there is $r \in R$ such that $r \Vdash_R \neg(K \models \varphi_1[s])$. We can assume that $r = 1_R$ and there is $\chi' \equiv (\bigwedge_{i \in I'} (z_i)^{\epsilon_i} = 0)$, such that in V, $K \models (\chi'[s])$, but R forces that $K \models \neg(\chi'[s])$. Suppose $s(z_i^{\epsilon_i}) = C'_i$. Denote $|I'| = k'$.

Let η be a R-name such that $\Vdash_R (\eta \subseteq \prod_{i<k'} C'_i) \wedge (|\eta| = \aleph_1) \wedge$ (every two element pair subset of η is OP). Let $\{\langle r_\alpha, b_\alpha\rangle \mid \alpha < \aleph_1\}$ be such that $\{b_\alpha \mid \alpha < \aleph_1\}$ is a family of pairwise disjoint k'-sequences, and $r_\alpha \in R$ such that $r_\alpha \Vdash_R \check{b}_\alpha \in \eta$. We clean the sequence $\{\langle r_\alpha, b_\alpha\rangle \mid \alpha < \aleph_1\}$ as much as possible, as appeared in other similar proofs (see theorem 2.1 for example).

We can assume that the r_α's are pairwise disjoint, and of the size l . Suppose $r_\alpha = \{\gamma(\alpha, 0), \cdots, \gamma(\alpha, l-1)\}$, where the $\gamma(\alpha, i)$'s appear in an increasing order. For $i < l$, let $a(\gamma(\alpha, i)) = a(\gamma(\alpha, i), 0)^\frown \cdots {}^\frown a(\gamma(\alpha, i), m-1)$, where $a(\gamma(\alpha, i), j)$ is the j element in $\gamma(\alpha, j)$. Recall that every such $a(\gamma(\alpha, i), j)$ is a k-tuple element.

Let $a_\alpha = a(\gamma(\alpha, 0))^\frown \cdots {}^\frown a(\gamma(\alpha, l-1))$, and let $c_\alpha = a_\alpha^\frown b_\alpha$. Then a_α is a concatenation of k-tuples, and b_α is a k'-tuple.

As done many times before, let F be the closure of $\{c_\alpha \mid \alpha < \aleph_1\}$ in $\Re^{l*m+k'}$, and let $\delta_0 \in C_M$ be such that F is definable in $N \preceq M$ where $|N| \cap \aleph_1 = \delta_0$. Let

α be such that $\delta_0 < c_\alpha$.

We want to duplicate $\langle r_\alpha, b_\alpha \rangle$, but before we start, we need additional notation. Let $d(i,j) = a(\gamma(\alpha,i),j)$, $d(i) = d(i,0)^\frown \cdots ^\frown d(i,m-1)$. Meaning that $d(i,j)$ is a k-tuple, and $d(i)$ is such that if $r_\alpha = \langle q_0, \cdots, q_{l-1} \rangle$ then $q_i \Vdash_Q d(i) \in \tau$. Let a, b, c denote respectively a_α, b_α and c_α.

Claim 4.3. *There are $c^0, c^1 \in F$ such that for $i = 0, 1$*

$$c^i = d^i(0,0)^\frown \cdots d^i(0,m-1)^\frown \cdots d^i(l-1,0)^\frown \cdots d^i(l-1,m-1)^\frown b^i$$

and (1) $\left\langle b^0, b^1 \right\rangle$ is not OP; (2) for every $i < l$, there is $j < m$, such that $\left\langle d^0(i,j), d^1(i,j) \right\rangle$ is not OP.

The contradiction follows easily from the duplication claim, by choosing c_β near enough to c^0, and c_γ near enough to c^1. For such β, γ, Because of demand (2) in the above claim, $r_\beta \cup r_\gamma \in R$, but due to demand (1), $\langle b_\beta, b_\gamma \rangle$ is not OP.

Proof. Let $\leq_l$ denote the lexicographic order on $l \times m$. Then, for $\langle i,j \rangle \in l \times m$, there are $\beta(i,j)$ in C_M such that $\beta(0,0) \leq d(0,0) < \beta(0,1) \leq, \cdots, \leq d(0,m-1) < \beta(1,0) \leq, \cdots, < d(l-1,m-1)$. Let $t = l * m$, and $\{\beta_i \mid i < t\}$ be the enumeration of the above $\beta(i,j)$'s. Let $E_j = [\beta_j, \beta_{j+1})$. In the duplication argument, there are two cases to consider for how b is divided.

case 1: All the elements of b are in the same E-slice, suppose E_j. Recall that for every b_i in b, $b_i \in C_i'$. As in V, there is no $C \in K$, such that, for every $i < k'$, $C \preceq C_i'$, then when we define the formula φ_j, in the duplication argument, we can duplicate b into two copies b', b'' such that $\langle b', b'' \rangle$ is not OP (using the well known argument which appears in theorem 2.1).

As $m \geq 2$, then for every $g < l$ there is $h \leq m$, such that $d(g, h) \in E_s$, and $E_s \neq E_j$. Hence in the duplication argument, in defining φ_s, we will duplicate $d(g, h)$ into two copies, $d(g, h)', d(g, h)''$ such that $\langle d(g, h)', d(g, h)'' \rangle$ is not OP.

When constructing the two required elements of F, c^0, c^1, stage after stage, we are free to choose any of the copies to construct c^0 with (with the other copy we will construct c^1).

Then we can assume that $b' \in c^0$, and $b'' \in c^1$, and as $\langle b', b'' \rangle$ is not OP, demand (1) of the claim is met; and for every $i < l$, there is $j < m$, such that $\langle d(i, j)', d(i, j)'' \rangle$ is not OP, and since, without loss of generality, $d(i, j)' \in c^0$, and $d(i, j)'' \in c^1$, then demand (2) of the claim is met as well.

case 2: Suppose that there are $i, j < k'$ such that $b_i \in E_v$, $b_j \in E_w$, where $E_v \neq E_w$.

In this case, if we are in step s and there are $g < l$, and $h < m$ such that $d(g, h) \in E_s$, we define φ_s such that $d(g, h)$ will be duplicated into two copies, $d(g, h)', d(g, h)''$ such that $\langle d(g, h)', d(g, h)'' \rangle$ is not OP; and if we are in step s where there is no such $d(g, h)$ in E_s, then we just duplicate any elements of b that might be in E_s into two copies.

We are now constructing the two required elements of F, c^0, c^1, stage after stage. Suppose that we are already past stage v, and are about to choose copies from stage w. Suppose b'_j, b''_j are the copies of b_j, and $b'_j < b''_j$. Then if in stage v, we chose such that $b^0_i < b^1_i$, we will now assign b'_j to c_1, and b''_j to c_0, so $b^1_j < b^0_j$. And if $b^1_i < b^0_i$, then we will make the assignments such that $b^0_j < b^1_j$.

Then $\left\langle b^0, b^1 \right\rangle$ is not OP, as $b^0_i < b^1_i$ if and only if $b^1_i < b^0_i$, so demand (1) of the claim is met. Note that, for every $i < l, j < m$, the copies of $d(i, l)$ are not OP, hence demand (2) is met as well. $\square$ 4.3

2) The reason that R is c.c.c follows from the same arguments as (1). We take an $\aleph_1$ conditions of R and by duplication we get two compatible conditions.

3) $\Vdash_R$ (Q is not c.c.c)

Let G be an R-generic filter, and suppose by contradiction, that after forcing with R, the set of elements that are stored in G, $(\bigcup(\bigcup(\bigcup G))$ to be more specific), is countable. Let D^G denote that set.

Then we will take all the $r \in R$, such that for each r there is an $\alpha < \aleph_1$ such that $r \Vdash (\alpha = sup(D^G))$. These r form a dense set, and they are all contradictive, thus form a maximal antichain in R. As R is c.c.c, this antichain is countable, and therefore there is $b \in \omega_1$, such that $\Vdash_R$ (all of D^G elements are smaller than b). But as for every $b \in \omega_1$, there is $x > b$, such that $x \in a$, for some $a \in q$ where $q \in Q$, and $q \in r'$ where $r' \in R$, it follows that for an R-generic filter G, such that $r' \in G$, $x \in D^G$, whereas $x > b$, a contradiction. Therefore there is an $r \in R$, such that $r \Vdash_R D^G$ is uncountable). Then $r \Vdash_R$ (Q is not c.c.c), and $R \restriction r$ is the forcing set we will use. $\square$ 4.2

5 The successor stage

We will now show the successive step of the induction, where ν's task is a P_ν-name of a member A of K^H.

We will find $\tau \subseteq n$, and a forcing set that will create a universe, in which $A \cong \bigcup_{i\in\tau} A_i$, and $\langle\{A_i \mid i < n\}, \preceq, *\rangle \cong \langle I(L), \preceq, *\rangle$. We thus state the following lemma, which is lemma 10.7 from [ARS], and its proof will occupy us through the whole section.

Lemma 5.1. *Let $\{A_i | i < n\}$, $\{B_i | i < n\}$ and $\{F_i | i < n\}$ be as defined before. For each $i < n$ let $B(i) \in F_i$ be such that $B(i^*) = B^*(i)$, and such that $K \models \varphi_1[B(0), \cdots, B(n-1), A_0, \cdots, A_{n-1}]$. Let $A \in K^H$.*

Then there is $\tau \subseteq n$, and $B'(i) \in F_i$ such that for every $i < n$, $B'^*(i) = B'(i^*)$, and there is an $\aleph_1$ c.c.c forcing set P such that

1. $\Vdash_P (A \cong \bigcup_{i\in\tau} A_i)$.
2. $\Vdash_P (K \models \varphi_1[B'(0), \cdots, B'(n-1), A_0, \cdots, A_{n-1}])$.

Note that as the induction continues, new members of K are created. However, as every member of K becomes a member of K^H before $\aleph_2$ steps, we can consider only the members of K^H, as the list of tasks we have made will take care of all new members of K^H as well.

The reminder of this section is devoted to the proof of the above lemma, but first we show how to define $\pi_{\nu+1}$ and $B(\nu+1, i)$, and how $(B) \to (A)$ of the main theorem, theorem 3.3, follows from what has been described so far. Let $\pi_{\nu+1}$ be the P_ν-name of the forcing set P of lemma 5.1 and $\check{B}(\nu+1, i)$ be the P_ν-name of the B'_i of lemma 5.1.

Let $P = P_{\aleph_2}$. Clearly $\Vdash_P MA_{\aleph_1}$, and $\Vdash_P (\forall A \in K^H)(\exists\tau \subseteq n)(A = \bigvee_{i\in\tau} A_i)$. It is already in V that if $i \preceq j$ then $A_i \preceq A_j$. So it remains to show that if $i \not\preceq j$

then $\Vdash_P A_i \not\preceq A_j$. Suppose the contrary. Let G be P-generic, $W = V[G]$, and let $f : A_i \to A_j$ be an OP function belonging to W. As $|f| = \aleph_1$, then for some $\nu < \aleph_2$, $f \in V[G \cap P_\nu] \overset{def}{=} W_\nu$. But in W_ν, $B(\nu, i) \perp A_j$, a contradiction, as $B(\nu, i) \subseteq A_i$, and $f(B(\nu, i)) \subseteq A_j$.

Hence $\langle \{A_i \,|\, i < n\}, \preceq \rangle \cong \langle I(L), \preceq, *, \rangle$. Now, using $MA_{\aleph_1}$, we get from Proposition 3.8 that $I(K^{HZ}) = \{A_i \,|\, i < n\}$, and that $\left\langle K^{HZ}, \preceq, * \right\rangle \cong \langle L, \preceq, * \rangle$. That settles theorem 3.3 $\square$.

We now turn to the proof of lemma 5.1.

Proof of lemma 5.1:

Let $A \in K^H$. We can assume that A is dense in $\Re$, and by theorem 1.13 we can assume that $A \in K^{2H}$ which means that every two intervals of A are order isomorphic. In fact we can assume that A is isomorphic to any non empty interval of A.

The proof of lemma 5.1 will be made in four steps:

1. We find $\tau \subseteq n$ such that A can be isomorphic to $\bigcup_{i \in \tau} A_i$. We will see that we do not have many options in choosing τ.

2. Let $F(i) = F_i \restriction B(i)$. It is easy to see that $F(i)$ is also a filter, and as $F_i^* = F_{i^*}$, and $B(i^*) = B^*(i)$, then $F(i^*) = F^*(i)$. From each $F(i)$ we choose residues of $B(i)$, that is $B_0(i) \in F(i)$, such that the structure, when considering the $B_0(i)$'s instead of the $B(i)$'s, will remain solid. Apart from that, we choose dense subsets of A: $\{B_1(i) \,|\, i \in \tau\}$ such that for every $i \in \tau$, replacing $B_0(i)$ with $B_1(i)$ will leave the structure solid as well.

3. We expand every $B_1(i)$ to $B_1^i \supseteq B_1(i)$, such that $A \subseteq \bigcup_{i \in \tau} B_1^i$, and expand every $B_0(i)$ to B_0^i, where for every $i < n$, $B_0(i) \subseteq B_0^i$, and $A_{\tau_i} \subseteq \bigcup_{j \in \tau_j} B_0^j$.

By doing that, we will have make sure the structure is maintained when considering the expanded sets.

4. For every $i \in \tau$ we trim B_1^i to a certain set D_1^i, and trim B_0^i to a certain set D_0^i, so that $\bigcup_{i\in\tau} D_0^i = \bigcup_{i\in\tau} A_i$, and $\bigcup_{i\in\tau} D_1^i = A$. Then we finally define the forcing set which will create a universe in which $\bigcup_{i\in\tau} D_0^i$ and $\bigcup_{i\in\tau} D_1^i$ are isomorphic, while keeping the structure maintained; thus meeting the requirements of lemma 5.1.

Our first goal will be to find a τ subset of n, that eventually A will become isomorphic to $\bigcup_{i\in\tau} A_i$.

5.1 Step 1 - Defining the collection $\{A_i \mid i \in \tau\}$

Recall that we may assume that $A \in K^{2H}$ and dense in $\Re$. Suppose that A has a subset $D \in K$ such that for some $i < n$, $D \preceq A_i$. When we want to find $\tau \subseteq n$ such that, eventually, A will become isomorphic to $\bigcup_{i\in\tau} A_i$, we have to take D into consideration, as if there is A_j which is far from A_i, then j cannot be in τ. This notion will lead us to the following:

Definition 5.2. *For $A \in K^H$, we define*

$$\eta(A) = \{\sigma \subseteq n \mid (\exists D \in K)((D \preceq A) \wedge (\forall i \in \sigma, D \preceq A_i) \wedge (\forall i \notin \sigma, D \perp A_i))\}.$$

The set D which satisfies the above conditions for $\sigma \in \eta(A)$ is denoted as a a witness for *σ.*

Corollary 5.3. *1. For every $\sigma \in \eta(A)$, if $i \in \sigma$ and $i \preceq j$ than $j \in \sigma$.*

2. For every $D_1, D_2 \in K$, if $D_1 \subseteq D_2$ then $\eta(D_1) \subseteq \eta(D_2)$.

3. For every $A \in K^H$, $\eta(A) \neq \emptyset$

Proof. Easy. $\square$

by 1.11 we can assume that $A \in K^{2H}$, and is isomorphic to any non empty interval of A; hence for every two non empty intervals, I, J, of A, $\eta(I) = \eta(J)$.

For $\sigma \in \eta(A)$, we denote $\bigcap_{i\in\sigma} \tau_i$ as the *set of roots of* σ. If $\sigma = \emptyset$ as can happen, we simply denote $\bigcap_{i\in\emptyset} \tau_i = n$.

Note that, if $i, j \in \sigma$ where $\sigma \in \eta(A)$, then there is $D \in K$ witness for σ such that $D \preceq A_i, A_j$. As our induction preserves the structure, it means that A_i and A_j were not far at the initial step of the induction, hence there is $k < n$ such that $A_k \preceq A_i, A_j$, meaning that $\tau_i \cap \tau_j \neq \emptyset$,. As σ is finite, and every τ_i is downward closed, the set of roots of σ is not empty, and is downward closed as well.

Definition 5.4. *Let $\tau = \{j \in n \mid (\exists\sigma \in \eta(A)$ such that j is a root of $\sigma\}$.*

If $\eta(A)$ is empty let τ simply be $\{1, \cdots, n\}$.

Note that τ is downward closed. By the discussion above it is natural why we wish A to become isomorphic to shuffle of $\bigcup_{i\in\tau} A_i$.

Example: Suppose $\{i, j\} = \eta(A)$ and let $D \preceq A$ be a witness for $\{i, j\}$. Then, although in the initial step, D would have had to be embedded in a root of τ_i, τ_j, it does not have to be the situation in the following steps . That is because the residues of the B_i's are getting smaller, and D can be embedded in $A_i \backslash \left(B(i) \cup \bigcup_{l\preceq i} A_l\right)$, and in $A_j \backslash \left(B(j) \cup \bigcup_{l\preceq j} A_l\right)$ which are no longer empty as they were in the initial step. Suppose $A_k \preceq A_i, A_j$. Then in this example k is a root of σ, so making A isomorphic to A_k will not harm the structure. However making A isomorphic to A_i can make A_i and A_j isomorphic, perhaps against our will.

5.2 Step 2 - Finding dense sets in A

Finding τ, we now proceed. This step is an elaboration of lemma 10.8 from [ARS].

Lemma 5.5. *(CH) Let $A_i, B(i), F(i), A$ and τ be as before. Then there are $\{B_0(i) \mid i < n\}$ and $\{B_1(i) \mid i \in \tau\}$ such that:*

1. *$B_0(i) \in F(i)$, and $B_0(i^*) = B_0^*(i)$.*

2. *$B_1(i) \in K$, $B_1(i)$ is a dense subset of A, and for $i \neq j$, $B_1(i) \cap B_1(j) = \emptyset$.*

3. *For every $i, j \in \tau$ where $i \neq j$, $B_1(i) \perp\!\!\!\perp B_1(j)$ and $B_1(i) \perp B_1^*(i)$.*

4. *for every $j \in \tau$, $i < n$, $B_1(j) \perp\!\!\!\perp B_0(i)$.*

Proof. Recall that we have defined τ as $\{j \in n \mid \exists \sigma \in \eta(A)$ such that j is a root of $\sigma\}$. For every $\sigma \in \eta(A)$ we choose a set D_σ which is a witness of σ. As A is dense in $\Re$, and isomorphic to every non empty interval of A, then for every rational interval I, there is D_σ^I such that $D_\sigma^I \subseteq I$, and $D_\sigma^I \cong D_\sigma$. Thus we define D_σ^m as the mixing of D_σ; that is $D_\sigma^m = \bigcup\{D_\sigma^I \mid I$ is a rational interval$\}$.

For every $j \in \tau$ let $D_1(j) = \bigcup\{D_\sigma^m \mid j$ is a root of $\sigma\}$. Then $D_1(j)$ is a finite union of such D_σ^m's and is dense in A. It is in $D_1(j)$ that we will find a dense subset $B_1(j)$ which will suit our needs.

Let $U \subseteq n$ be such that for every $i < n$, $|U \cap \{i, i^*\}| = 1$. We use lemma 1.14 with $B(i)$, with its enumeration, instead of A_i for every $i \in U$, and $D_1(j)$ as A'_j for every $j \in \tau$. We obtain $\{B_0(i) \mid i \in U\} \subseteq K$ where every $B_0(i)$ is a dense subset of $B(i)$, and pairwise disjoint $\{B_1(i) \mid i \in \tau\} \subseteq K$ where every $B_1(i)$ is a dense subset of $D_1(i)$ (and therefore dense in A). For every $i \in U$, if $i \neq i^*$, we define $B_0(i^*) = B_0^*(i)$; as for every $i < n$, $B(i^*) = B^*(i)$, then $B_0(i)$ is well defined and is dense in $B(i)$. Apart from that, for every $i, j \in \tau$ where $i \neq j$, $B_1(i) \perp B_1^*(i)$, $B_1(i) \perp\!\!\!\perp B_1(j)$, and for every $j \in \tau$ and $i \in U$, $B_1(j) \perp\!\!\!\perp B_0(i)$, and as $B_0(i^*) = B_0^*(i)$ then $B_1(j) \perp\!\!\!\perp B_0(i)$ for every $j \in \tau$ and $i < n$ as well.

It remains to show that for every $i < n$, $B_0(i) \in F(i)$. As every $B(i)$ is dense in $\Re$, then we can easily construct the $B_0(i)'s$ in lemma 1.14 such that if $\{b(\alpha, i, I) | \alpha < \aleph_1\}$ is the enumeration of every interval of $B(i)$, then $B_0(i)$ contains a club of these elements. Then for every $i \in U$ $B_0(i) \in F(i)$, and as $F(i^*) = F^*(i)$, then $B_0(i) \in F(i)$ for every $i < n$. $\square$

Finding $\{B_1(i) \mid i \in \tau\}$, and $\{B_0(i) \mid i < n\}$, we proceed. We already know that $K \models \varphi_1[B(0), \cdots, B(n-1), A_0, \cdots, A_{n-1}]$. First we wish to see that $K \models \varphi_1[B_0(i)|i < n; A_0, \cdots, A_{n-1}]$, where every x_i is replaced by $B_0(i)$, and every y_i is replaced by A_{τ_i}. Then we wish to see that for every $i \in \tau$ we can sometimes use $B_1(i)$ as an assignment for x_i instead of $B_0(i)$, and that K will still suffice φ_1 with this new assignment. Later, this fact will insure us that even if $B_0(i)$ and $B_1(i)$ become close, our structure will remain solid. To be more accurate, we define the following:

Definition 5.6. *For a variable x_i, x_i^0 and x_1^1 are called* copies *of x_i, and for a variable y_τ, y_τ^0, and y_τ^1 are called* copies *of y_τ.*

A formula φ' is called a copy *of φ_0 if it is gotten from φ_0 by a syntaxis replacement of every occurrence of a variable in φ_0 by one of it's copies (two occurrences of the same variable can be replaced by two different copies).*

Let ψ_0 be the conjunction of all copies of φ_0, and ψ_1 be the conjunction of all copies of φ_1.

We again make the convention that for an assignment s such that $Dom(s) \subseteq \{x_i^l \,|\, i < n,\, l \in \{0,1\}\} \cup \{y_\tau^l \,|\, \tau \subseteq n,\, l \in \{0,1\}\}$, $K \models \psi[s]$ means that all conjuncts of ψ_t which variables belong to $Dom(s)$ are satisfied. Note that to obtain a copy of φ_1, we first take φ_1, and only then replace the variables with their copies, and make the assignment. Otherwise there could be additional farness formulas in ψ as the assignment now includes A_τ as well.

Lemma 5.7. *(CH) Let τ, $\{B_0(i) \mid i < n\}$, $\{B_1(i) \mid i \in \tau\}$, $\{A_i \mid i < n\}$ and A be as above. Let $A_{\tau_i}^0 = A_i$, $A_\tau^1 = A$, $B_i^0 = B_0(i)$, $B_i^1 = B_1(i)$. Then*

$$K \models \psi_1[B_i^0 | i < n; B_j^1 | j \in \tau; A_{\tau_i}^0 | i < n; A_\tau^1],$$

where each B_i^0 replaces x_i^0, B_i^1 replaces x_i^1, $A\tau_i{}^0$ replaces $y_{\tau_i}^0$, and A_τ^1 replaces y_τ^1.

Proof of lemma 5.7.

Denote s' as the assignment mentioned in the lemma. We want to see that $K \models \psi_1[s']$. To prove that, we will go over all the F-formulas, and see that for every F-formula χ, if $K \models \neg\chi[s']$ then χ is not a conjunct of ψ_1. This task appears tedious, but we will show that there are only several types of F-formulas that will interest us, as other types will follow automatically.

Definition 5.8. *Given an F-formula* $\chi \equiv (\bigwedge_{i\in I} z_i^{\epsilon_i} = 0)$*, we denote* $\chi^* \stackrel{def}{=} (\bigwedge_{i\in I} z_i^{1-\epsilon_i} = 0)$. *Similarly let* χ^+ *be the formula obtained from a copy of* χ*, by replacing every occurrence of* $(x_i^0)^*$ *or* $(y_{\tau_i}^0)^*$ *with* $x_{i^*}^0$ *and* $y_{\tau_i^*}^0$ *respectively.*

Claim 5.9. *Let* $\chi \equiv (\bigwedge_{i\in I} z_i^{\epsilon_i} = 0)$ *be an F-formula. Let* χ' *be an F-formula obtained from* χ *such that for every* $i \in I$*, some (or none) occurrences of* x_i^* *are replaced with* x_{i^*}*, and some (or none) occurrences of* $y_{\tau_i}^*$ *are replaced with* $y_{\tau_i^*}$*. Then* χ' *is a conjunct of* φ_1 *if and only if* χ *is a conjunct of* φ_1 *(recall that* $\tau_i^* = \{j^* \mid j \in \tau_i\}$ *is actually* τ_{i^*}*,* $A \preceq B$*).*

Proof. As all the *fine-assignments* (defined in subsection 3.4) has the property that if s_0 is such an assignment then $s_0(x_i^*) = s_0(x_{i^*})$, and $s_0(y_{\tau_i}^*) = s_0(y_{\tau_i^*})$, then in every finite distributive lattice $L' \models \chi[s_0]$ if and only if $L' \models \chi'[s_0]$.

Lemma 5.10.

1. *χ is a conjunct of ψ_1 if and only if χ^* is a conjunct of ψ_1 if and only if χ^+ is a conjunct of ψ_1.*
2. *$K \models \chi[s']$ if and only if $K \models \chi^*[s']$ if and only if $K \models \chi^+[s']$.*

Proof.

1. See lemma 3.14 for χ^*. By the definition of ψ_1, and the above claim, $\chi \in \psi_1$ if and only if $\chi^+ \in \psi_1$.
2. As for every $i < n$ $(B_i^0)^* = B_{i^*}^0$, and $(A_i^0)^* = A_{\tau_i^*}^0$, then $K \models \chi[s']$ if and only if $K \models \chi^+[s']$. It is easy to see that $K \models \chi[s']$ if and only if $K \models \chi^*[s']$.

□

Hence we see that most of the types of the F-formulas are already sorted out. In fact the F-formulas that we need to inspect closely are only those of the form

$$(\bigwedge_{i\in\sigma} y^0_{\tau_i}) \wedge (\bigwedge\{t \mid t \in T\}) = 0,$$

where σ is a set of variables, and T is subset of

$$\{x^0_j \mid j < n\} \cup \{x^1_j \mid j \in \tau\} \cup \{(x^1_j)^* \mid j \in \tau\} \cup \{y^1_\tau, (y^1_\tau)^*\}$$

and T intersects the union of the first three sets at most once.

We will now go over the possibilities of T, and see that for every χ of that form, if $K \models \neg\chi[s']$ then χ is not a conjunct of ψ_1. Note that it is interesting to see that if y^1_τ is taking place in a copy of a farness formula, then there is $j < n$ such that $\tau = \tau_j$.

Case 1 $T = \{x^0_j\}$ for some $j < n$.

Suppose $\chi \equiv ((\bigwedge_{i\in\sigma} y^0_{\tau_i}) \wedge x^0_j = 0)$, and $K \models \neg\chi[s']$ As B^0_j is a residue of $B(j)$, and every F-formula that is sufficed with $B(j)$, will be sufficed with a residue of $B(j)$, then χ is not a conjunct of ψ_1.

Case 2 $T = \{x^1_j\}$ for some $j < n$.

Suppose $\chi \equiv ((\bigwedge_{i\in\sigma} y^0_{\tau_i}) \wedge x^1_j = 0)$, and $K \models \neg\chi[s']$. Then there is $C \in K$ such that $C \preceq A^0_{\tau_i}$ for every $i \in \sigma$, and $C \preceq B^1_j$. As $B^1_j \subseteq D_1(j)$, then $C \preceq D_1(j)$. Hence, by the definition of $D_1(j)$, there is $D \in K$, a witness for $\sigma' \subseteq n$ such that $j \in \bigcap_{l\in\sigma'} \tau_l$ and $C \preceq D$. As D witnesses σ', D is far from all A_{τ_l} where $l \notin \sigma'$. Since ,for every $i \in \sigma$, $D \wedge A_{\tau_i} \neq 0$, it follows that $\sigma \subseteq \sigma'$. So $j \in \bigcap_{i\in\sigma} \tau_i$. Hence, by lemma 3.14, $x_j \wedge \bigwedge_{i\in\sigma} y_{\tau_i} = 0$ is not an F-formula derived from φ_0; therefore χ is not a conjunct of ψ_1.

Case 3 $T = \{(x^1_j)^*\}$ for some $j < n$.

As in previous cases it follows that $K \models \neg\chi[s']$ where $\chi \equiv ((\bigwedge_{i\in\sigma} y^0_{\tau_i}) \wedge (x^1_j)^* = 0)$, and again, it means that $(B^1_j)^* \wedge \bigwedge_{i\in\sigma} A^0_{\tau_i} \neq 0$. The following claim will show that $j^* \in \bigcap_{i\in\sigma} \tau_i$, and, precisely as in case 2, it implies that χ is not a conjunct of ψ_1.

Claim 5.11. *If* $(B^1_j)^* \wedge \bigwedge_{i\in\sigma} A^0_{\tau_i} \neq 0$ *then* $j^* \in \bigcap_{i\in\sigma} \tau_i$.

Proof. Suppose $(B^1_j)^* \wedge \bigwedge_{i\in\sigma} A^0_{\tau_i} \neq 0$. Then ,as before, there is $C \in K$ such that $C \preceq A_{\tau_i}$ for every $i \in \sigma$, and $C \preceq (B^1_j)^*$ which leads to $C^* \preceq D_1(j)$. So there is $\sigma' \in \eta(A)$ such that $j \in \bigcap_{i\in\sigma'} \tau_i$, and there is $D \in K$, a witness for σ', such that $D^* \wedge \bigwedge_{i\in\sigma} A^0_{\tau_i} \neq 0$.

So $D \wedge \bigwedge_{i\in\sigma} (A^0_{\tau_i})^* \neq 0$, and because $(A^0_{\tau_i})^* = A^0_{\tau_{i^*}}$, then $D \wedge \bigwedge_{i\in\sigma} A^0_{\tau_{i^*}} \neq 0$, hence $D \wedge \bigwedge_{i\in\sigma^*} (A^0_{\tau_i}) = 0$ where $\sigma^* = \{j^* \mid j \in \sigma\}$. Hence, $\sigma^* \subseteq \sigma'$. So $j \in \bigcap_{i\in\sigma^*} \tau_i$ leads to $j^* \in (\bigcap_{i\in\sigma^*} \tau_i)^* = \bigcap_{i\in\sigma^*} (\tau_i)^* = \bigcap_{i\in\sigma^*} \tau_{i^*} = \bigcap_{i\in\sigma} \tau_i$. $\square$

Case 4 $T = \{y^1_\tau\}$.

As in previous cases, the meaning is $\bigwedge_{i\in\sigma} A_i \wedge A \neq 0$. So there is $\sigma' \in \eta(A)$, such that $\sigma \subseteq \sigma'$. So $\bigwedge_{i\in\sigma'} A_i \neq 0$, and since $K \models \varphi_1[A_{\tau_0}, \cdots, A_{\tau_{n-1}}]$, it follows that $\bigcap_{i\in\sigma'} \tau_i \neq 0$ as well. So there is $j \in n$ such that $j \in \bigcap_{i\in\sigma'} \tau_i$. As $\sigma \subseteq \sigma'$, it means that $j \in \bigcap_{i\in\sigma} \tau_i$ as well, and by the definition of τ, $j \in \tau$; hence χ is not a conjunct of ψ_1.

Case 5 $T = \{(y^1_\tau)^*\}$.

This case can be easily reduced to case 4, by replacing χ by χ^*, and then with $(\chi^*)^+$.

Case 6 $T = \{(x^1_j)^*, y^1_\tau\}$.

Claim 5.12. *if* $(B^1_j)^* \wedge A^1_\tau \neq 0$ *then* $j^* \in \tau$.

Proof. Suppose $(B^1_j)^* \wedge A^1_\tau \neq 0$. Let $\sigma \in \eta(A)$ be such that $j \in \bigcap_{i\in\sigma} \tau_i$, and let $D \in K$ be such that $D \preceq D^*_\sigma \wedge A^1_\tau$. As $D^* \preceq D_\sigma$, then for every $i \in \sigma^*$, $D \preceq A_i$, and for every $i \notin \sigma^*$ $D \wedge A_i = 0$; and as $D \preceq A$, then D is a witness for σ^*. Hence, $\sigma^* \in \eta(A)$, and since $j \in \bigcap_{i\in\sigma} \tau_i$, then $j^* \in \bigcap_{i\in\sigma^*} \tau_i$, meaning j^* is a root of σ^*. So by the definition of τ, $j^* \in \tau$. $\square$

Now, as in previous cases, $T = \{(x^1_j)^*, y^1_\tau\}$ means $\bigwedge_{i\in\sigma} A_i \wedge (B^1_j)^* \wedge A \neq 0$. As $(B^1_j)^* \wedge A \neq 0$, then by the above claim $j^* \in \tau$, and as $\bigwedge_{i\in\sigma} A_i \wedge (B^1_j)^* \neq 0$, then because of claim 5.11, it follows that $j^* \in \bigcap_{i\in\sigma} \tau_i$. So all in all we get $j^* \in \tau \cap (\bigcap_{i\in\sigma} \tau_i)$, which means that χ is not a conjunct of ψ_1.

Case 7 $T = \{(y^1_\tau)^*, y^1_\tau\}$.

As in previous cases $\bigwedge_{i\in\sigma} A_i \wedge A^* \wedge A \neq 0$. Then let $D \in K$ be such that $D \preceq A, \bigwedge_{i\in\sigma} A_i$, and $D \preceq A^*$. As $D \preceq \bigwedge_{i\in\sigma} A_i$, there is $\sigma' \in \eta(A)$ such that $\sigma \subseteq \sigma'$, and D witnesses σ'; hence there is $j < n$ such that $j \in \bigcap_{i\in\sigma'} \tau_i$, so $j \in \tau$. As $D \preceq A^*$,then $D^* \preceq A$ and as in previous cases, $\sigma^* \in \eta(A)$, hence $j^* \in \bigcap_{i\in\sigma'^*} \tau_i$. So $j^* \in \tau$ (as j^* is a root of σ^*), hence $j \in \tau^*$. Then χ is not a conjunct of ψ_1, as $j \in (\tau \cap \tau^* \cap \bigcap_{i\in\sigma} \tau_i)$.

Case 8 $T = \{x^0_i, y^1_\tau\}$

This case resembles case 4. if $\chi \equiv ((\bigwedge_{i\in\sigma} y^0_{\tau_i}) \wedge x^0_i \wedge y^0_\tau = 0)$, means $\bigwedge_{i\in\sigma} A_i \wedge B^0_i \wedge A \neq 0$, then it must be that $i \in \sigma$.

If $\chi \equiv (x^0_i \wedge y^0_\tau = 0)$, means $B^0_i \wedge A \neq 0$ then there is $\sigma' \in \eta(A)$, such that i is a root of σ', hence $i \in \tau$, and we reach a contradiction.

Case 9 $T = \{x_i^1, y_\tau^1\}$

Suppose $\chi \equiv (x_i^1 \wedge y_\tau^1 = 0)$, and $B_1(i) \wedge A \neq 0$. By definition, $B_1(i) \subseteq D_1(i)$, where $i \in \tau$. so χ is not a conjunct of ψ_1.

Because of the same reason, if $\chi \equiv ((\bigwedge_{i\in\sigma} y_{\tau_i}^0) \wedge x_i^1 \wedge y_\tau^1 = 0)$, means $\bigwedge_{i\in\sigma} A_i \wedge B_1(i) \wedge A \neq 0$, then the case is reduced to case 2, where $\bigwedge_{i\in\sigma} A_i \wedge B_1(i) \neq 0$.

The few other cases are proven in the very same way. That sums the proof for lemma 5.7. $\square$

5.3 Step 3 - Expanding the dense sets

The following section is an elaboration of lemma 10.9 from [ARS].

So far we have established a collection of pairwise disjoint dense subsets of A, $\{B_1(i) \mid i \in \tau\} \subseteq K$, and a collection of sets $\{B_0(i) \mid i < n\} \subseteq K$ where for each $i < n$, $B_0(i) \in F(i)$. What we seek, is a way to enlarge the $B_1(i)$'s so that the $B_1(i)$'s will include all A, and to enlarge the $B_0(i)$'s so that eventually the $B_0(i)$'s will include all $\{A_i \mid i \in \tau\}$. This, without causing the structure to change.

to simplify, suppose $|\tau| = n_1$; as it will not matter, we can rearrange $\{0, \cdots, n-1\}$, thus making $\tau = \{0, \cdots, n_1 - 1\}$.

Lemma 5.13. *(CH) Let $\{A_i \mid i < n\}$, A, $\{B_0(i) \mid i < n\}$, and $\{B_1(i) \mid i < n_1\}$ be as from lemma 5.5.*

Then there are pairwise disjoint $\{B'^0_i \mid i < n\} \subseteq K$, and pairwise disjoint $\{B'^1_i \mid i < n_1\} \subseteq K$, such that:

1. *For every $i < n$, $B_0(i) \subseteq B'^0_i$, and for every $i < n$, $A_{\tau_i} \subseteq \bigcup_{j\in\tau_i} B'^0_j$.*

2. *For every $i < n_1$, $B_1(i) \subseteq {B'}^1_i$, and for every $i < n_1$, $A \subseteq \bigcup_{j \in \tau} {B'}^1_j$.*

3. *Let $A^0_{\tau_i} = A_{\tau_i}$, and let $A^1_\tau = A_\tau$. Then, with the obvious assignment,*

$$K \models \psi_1[{B'}^0_i | i < n; {B'}^1_i | i < n_1; A^0_{\tau_i} | i < n; A^1_\tau].$$

Proof. Let M be a model, such that $|M| = \aleph_1$, all the information stored in the lemma is encoded in M, and M includes an enumeration of all the maximal real monotonic function (these functions can be either OP or OR). For every C_M-slice E, we decide how to divide the elements of E among the ${B'}^0_i$'s, and the ${B'}^1_i$'s . We will see that we can do that independently for every C_M-slice. We make an abuse of notation by defining the elements of E as the actual elements of the sets mentioned in the lemma.

Let $E = \{a_m \mid m \in \omega\}$ be a C^M-slice, and let F^E be the set of all maximal real monotonic functions which are definable, by the enumeration above, from the ordinals $\{\alpha \mid \alpha < min(E)\}$; we denote F^E as F. As C_M is the club of elementary substructures, it is clear that F^E is countable and closed under composition, and for every $f \in F$, f^* is in F as well.

We decide by induction on $m < \omega$ to which ${B'}^0_i$, or ${B'}^1_i$, a_m will eventually belong. At step m of the induction, we denote the so far expansions of the $B_0(i)$'s, and the $B_1(i)$'s as ${B'}^0_i(m)$, and ${B'}^1_i(m)$ respectively.

Claim 5.14. *Let M, C_M, E be as above. Let $C_0, \cdots, C_{k-1}$ be members of K, dense in $\Re$, such that $C_0, \cdots, C_{k-1}$ are encoded in M and such that there is no $C \in K$, such that for every $i < k$, $C \preceq C_i$. Then there are no $a \in E^C$, and $f_0, \cdots, f_{k-1}$ functions in F such that $f_0, \cdots, f_{k-1}$ are OP functions, and such that for every $i < k$, $f_i(a) \in C_i$.*

Proof. Suppose there were such functions $f_0, \cdots, f_{k-1}$ in F , And let $N \prec M$ be an elementary substructure such that $N \cap \aleph_1 = min(E^C)$. Let $\varphi(x)$ be the following formula:

$$\varphi(x) \equiv (\forall i < k) f_i(x) \in C_i$$

Then $M \models \exists x \varphi(x)$. As $a \notin N$, it follows that for every $\beta \in N$, $M \models (\exists x > \beta)\varphi(x)$. So $N \models (\forall y \exists x > y)\varphi(x)$. As $N \prec M$, $M \models (\forall y \exists x > y)\varphi(x)$. Hence the set $C' \equiv \{a \in M \mid (\forall i < k) f_i(a) \in C_i\}$ is uncountable. As C' contains a subset $C \in K$ (see 1.1) we reach a contradiction, as for every $i < k$ $f_i \restriction C$ is an OP function from C to C_i. $\square$

For every $m < \omega$, let s^m be the obvious assignment for stage m, that is an assignment which respectively puts for every $i < n$, $j < n_1$, ${B'}^0_i(m)$ in x^0_i, $({B'}^0_i(m))^*$ in $(x^0_i)^*$, ${B'}^1_i(m)$ in x^1_i, $({B'}^1_i(m))^*$ in $(x^1_i)^*$; and over the A_i's it puts for every $i < n$ $A^0_{\tau_i}$ in $y^0_{\tau_i}$, $(A^0_{\tau_i})^*$ in $(y^0_{\tau_i})^*$, A^1_τ in y^1_τ, and $(A^1_\tau)^*$ in $(y^1_\tau)^*$.

As our desire is to keep the farness between the upcoming sets, according to claim 5.14, our induction hypothesis will be the following:

Suppose we are at stage m, and let χ be a conjunct of ψ_1, where χ is of the form $\chi \equiv \bigwedge_{i \in I} (z^{\eta_i}_i)^{\epsilon_i} = 0$, where $\eta_i, \epsilon_i \in \{0, 1\}$ such that if $\epsilon_i = 1$ then $(z^{\eta_i}_i)^{\epsilon_i} = (z^{\eta_i}_i)^*$, and if $\epsilon_i = 0$ then $(z^{\eta_i}_i)^{\epsilon_i} = z^{\eta_i}_i$.

Then there is no $a \in E^C$, for which there are sets of OP functions $\{f_i \in F \mid i \in I\}$, such that for every $i \in I$, $f_i(a) \in s^m((z^{\eta_i}_i)^{\epsilon_i})$.

By claim 5.14, If there were such functions for some $a \in E^C$, χ could not have been a conjunct of ψ_1.

For step 0, let ${B'}^0_i(0) = B_0(i)$, ${B'}^1_i(0) = B_1(i)$. Then the induction hypothesis holds as

$$K \models \psi_1[B_i^0|i<n; B_i^1|i<n_1; A_{\tau_i}^0|i<n; A_\tau^1].$$

For step $m+1$, suppose ${B'}_0^0(m)\cdots{B'}_{n-1}^0(m)$, ${B'}_0^1(m)\cdots{B'}_{n_1-1}^1(m)$ have been defined. Our goal is to find to which of the ${B'}_i^0(m)$'s, ${B'}_i^1(m)$'s attach a_m, and while doing that, to keep the induction hypothesis.

If for every $i<n$, $a_m \notin A_{\tau_i}$, and $a_m \notin A$, then we make no changes; means that we define ${B'}_i^0(m+1) = {B'}_i^0(m)$ for $i<n$, and ${B'}_i^1(m+1) = {B'}_i^1(m)$ for $i<n_1$, and proceed to the next step on induction.

Otherwise, let
$I^0 = \bigcap\{\tau_i \subseteq n \mid (\exists f \in F) f \text{ is OP, and } f(a_m) \in A_{\tau_i}^0\}$.
$I^1 = \tau$ - if there is $f \in F$ such that f is OP, and $f(a_m) \in A_\tau^1$, and $\emptyset$ else.
$I^2 = \tau$ - if there is $f \in F$ such that f is OP, and $f(a_m) \in (A_\tau^1)^*$, and $\emptyset$ else.

For a non empty set X and a set Y let $X \Cap Y$ be the following operation: If $Y=\emptyset$ then let $X \Cap Y = X$. Else let $X \Cap Y = X \cap Y$.

Now let $\tau' \stackrel{def}{=} I^0 \Cap I^1 \Cap I^2$. The cases of lemma 5.7 show that τ' is not empty.

For example, suppose $i<n$ such that $a_m \in A_{\tau_i}^0 \setminus \bigcup_{j \preceq i} A_{\tau_j}^0$. Then the identity function is an OP function in F which puts a_m in $A_{\tau_i}^0$. So $\tau' \subseteq \tau_i$. For the same reason, if $a_m \in A_\tau^1$, then $\tau' \subseteq \tau$.

We next define:
$J^0 = \{i<n \mid (\exists f \in F), f \text{ is OP, and } f(a_m) \in {B'}_i^0(m)\}$
$J^1 = \{i<n_1 \mid (\exists f \in F), f \text{ is OP, and } f(a_m) \in {B'}_i^1(m)\}$
$J^2 = \{i<n \mid (\exists f \in F), f \text{ is OP, and } f(a_m) \in ({B'}_i^0(m))^*\}$
$J^3 = \{i<n_1 \mid (\exists f \in F), f \text{ is OP, and } f(a_m) \in ({B'}_i^1(m))^*\}$

Note that $J^0 \cup J^1$ contains at most one element. Else, suppose there were $i \in J^0, j \in J^1$, $i \neq j$; then by the induction hypothesis, $\chi_0 \equiv (x_i^0 \wedge x_j^1 = 0)$ would not be a conjunct of ψ_1, a contradiction, as χ_0 is a copy of a conjunct of conjunct of ψ_0. For the same reason $J^2 \cup J^3$ contains at most one element, and if there are $i \in J^0 \cup J^1$, and $j \in J^2 \cup J^3$, then $i = j^*$.

Note that if $i \in J^0 \cup J^1$, then $i \in \tau'$. That is because suppose that, without loss of generality $i \in J^0$, then ${B'}_i^0(m) \wedge \bigwedge_{\tau_i \in I^0} A_{\tau_i} \wedge A \wedge A^* \neq 0$, and as in case 7 of lemma 5.7, it follows that $i \in \bigcap_{\tau_i \in I^0} \tau_i \cap \tau \cap \tau^*$. Of course I^0, I^1 or I^2 maybe empty, but as seen before we get the same result.

For the same reason, if $i \in J^2 \cup J^3$ then $i^* \in \tau'$.

We finally come to the definition the member i_0, such that a_m will be added to ${B'}_{i_0}^0(m)$ or to ${B'}_{i_0}^1(m)$.

If $J^0 \cup J^1 \neq \emptyset$ then let $i = J^0 \cup J^1$, and we define $i_0 = i$. Else, if $J^2 \cup J^3 \neq \emptyset$ then let $i = J^2 \cup J^3$, and we define $i_0 = i^*$. Else let i_0 be any member of τ' (recall that $\tau' \neq \emptyset$).

Now for $i \neq i_0$, define $B_i^0(m+1) = B_i^0(m)$. If there is $j < n$ such that $a_m \in A_{\tau_j}^0$, define $B_{i_0}^0(m+1) = B_{i_0}^0(m) \cup \{a_m\}$; else, define $B_{i_0}^0(m+1) = B_{i_0}^0(m)$.

Same, for $i \neq i_0$, define $B_i^1(m+1) = B_i^1(m)$. If $a_m \in A\tau^1$, define $B_{i_0}^1(m+1) = B_{i_0}^1(m) \cup \{a_m\}$; else, define $B_{i_0}^1(m+1) = B_{i_0}^1(m)$.

It is left to see that the induction hypothesis holds. Let χ be a conjunct of ψ_1, where χ is of the form of $\chi \equiv (\bigwedge_{i \in I} (z_i^{\eta_i})^{\epsilon_i} = 0)$. Let s^{m+1} be the obvious assignment as indicated in the induction hypothesis. There are two cases which might disturb the induction hypothesis(which so far was being kept):

Case 1: There is a set of OP functions $\{f_i \in F \mid i \in I\}$, such that for every $i \in I$, $f_i(a_m) \in s^{m+1}((z_i^{\eta_i})^{\epsilon_i})$.

In the induction step, a_m was added to a certain $B'^0_{j_0}(m)$ or $B'^1_{j_0}(m)$. By the definition of τ' in the construction process $j_0 \in \tau'$, hence if z_i is of the sort of $(x_j)^{\epsilon_j}$,then $j^{\epsilon_j} = j_0$. This is because we have seen that if there is an OP function f such that $f(a_m) \in s(x_j^{\epsilon_j})$ then $j = j_0$. For the same reason, for every $i \in I$, if z_i is of the sort $y_\tau^{\epsilon_\tau}$ where τ is any subset of n, then $\tau' \subseteq \tau^{\epsilon_\tau}$, hence $j_0 \in \tau^{\epsilon_\tau}$.

So let $\chi' \equiv (\bigwedge_{i \in I} z_i^{\epsilon_i} = 0)$ be the origin of χ (means that χ is a copy of χ'). For every $i \in I$, if z_i is of the sort x_j, then $j = j_0$, and if z_i is of the sort y_τ where $\tau \subseteq n$, then $j_0 \in \tau$. Hence by lemma 3.14, χ' is not a conjunct of φ_1, so χ is not a conjunct of ψ_1, a contradiction.

Case 2: There is $a \in E^C$, $a \neq a_m$, and there is a set of OP functions $\{f_i \in F \mid i \in I\}$, such that for every $i \in I$, $f_i(a) \in s^{m+1}((z_i^{\eta_i})^{\epsilon_i})$.

Then there is $i_0 \in I$, such that $s^{m+1}((z_{i_0}^{\eta_{i_0}})^{\epsilon_{i_0}}) = s^m((z_{i_0}^{\eta_{i_0}})^{\epsilon_{i_0}}) \cup \{a_m\}$, and such that $\langle a, a_m \rangle \in f_{i_0}$. Else, for every $i \in I$, $f_i(a) \in s^m((z_i^{\eta_i})^{\epsilon_i})$, a contradiction to the induction hypothesis.

As f_{i_0} is an OP function in F, $f_{i_0}^{-1}$ is an OP function in F, and $\langle a_m, a \rangle \in f_{i_0}^{-1}$. As F is closed under compositions, $\{f_{i_0}^{-}1 \circ f_i \mid i \in I\}$ is a set of OP functions in F, and for every $i \in I$, $f_i(f_{i_0}^{-}1(a_m)) \in s^{m+1}((z_i^{\eta_i})^{\epsilon_i})$; a contradiction to case 1.

Now, for every $i < n$, define $B'^0_i = \bigcup_{E \in C^M} \left(\bigcup_{\{m \in \omega \mid a_m \in E\}} B_i^0(m) \right)$, and for every $i < n_1$, define $B'^1_i = \bigcup_{E \in C^M} \left(\bigcup_{\{m \in \omega \mid a_m \in E\}} B_i^1(m) \right)$.

We have managed to establish pairwise disjoint sets $\{B'^0_i \mid i < n\}$ such that for every $i < n$, $B_0(i) \subseteq B'^0_i$, and pairwise disjoint sets $\{B'^1_i \mid i < n_1\}$ such that for every $i < n_1$, $B_1(i) \subseteq B'^1_i$. As for every $i < n$, $|B^0_i| = \aleph_1$, and $B_0(i)$ is in K and dense in $\Re$, then B'^0_i is also in K, and dense in $\Re$, and same goes for the B'^1_i's as well.

By the construction process, if $a_m \in A_{\tau_i}$, then there is $j < n$ such that $a_m \in B'^0_j$ where $j \in \tau_i$, which insures that $A_{\tau_i} \subseteq \bigcup_{j\in\tau_i} B'^0_j$. Same, if $a_m \in A_\tau$, then there is $j < n_1$ such that $a_m \in B'^1_j$ where $j \in \tau$, which insures that $A_\tau \subseteq \bigcup_{j\in\tau_i} B'^1_j$.

Last, let s' be the sort of assignment, defined many times before, which refers to the B'^0_i's, B'^1_i's ,and let $\chi \equiv (\bigwedge_{i\in I}(z_i^{\eta_i})^{\epsilon_i} = 0)$ be a conjunct of ψ_1. Suppose by contradiction that $K \models \neg\chi[s']$. Then there is $D \in K$ such that for every $i \in I$, $D \preceq s'((z_i^{\eta_i})^{\epsilon_i})$. We can assume that there is $i_0 \in I$ such that $D \subseteq s'((z_{i_0}^{\eta_{i_0}})^{\epsilon_{i_0}})$. Then for every $i \in I$ there is an OP function f_i such that for every $d \in D$, $f_i(d) \in s'((z_i^{\eta_i})^{\epsilon_i})$. So we start from a model $N \preceq M$ in which all the f_i's are defined, and since D is unbounded, there is a C^M-slice E, and $a \in E$, such that $a \in D$, and for every $i \in I$, f_i is a member of F^E. Suppose $E = \{a_m \,|\, m \in \omega\}$, then there is $m < \omega$ such that $a = a_m$. Let s^m be the assignment defined at the slice E, at step m (as defined in the construction process). Then by the definition of s', for every $i \in I$, $a_m \in s^{m+1}((z_i^{\eta_i})^{\epsilon_i})$. A contradiction to the induction hypothesis for step $m+1$ at the C^M-slice E, as there are $a \in E$ and a set of OP functions $\{f_i \in F \mid i \in I\}$, such that for every $i \in I$, $f_i(a) \in s^{m+1}((z_i^{\eta_i})^{\epsilon_i})$, where $\chi \equiv (\bigwedge_{i\in I}(z_i^{\eta_i})^{\epsilon_i} = 0)$ is a conjunct of ψ_1. $\square$5.13

5.4 Step 4 - Defining the forcing set

After obtaining pairwise disjoint $\{B'^0_i \mid i < n\}$ and $\{B'^1_i \mid i < n_1\}$ from lemma 5.13, we now proceed. First, we would like to trim the B'^0_i's, and the B'^1_i's so that the union of the trim sets will be $\bigcup\limits_{i\in\tau} A_{\tau^0_i}$, and A^1_τ respectively. Then, we would finally like to state the forcing set, that will eventually lead to a universe where $\bigcup\limits_{i\in\tau} A_{\tau^0_i} \cong A$, and still $\langle\{A_i \mid i < n\}, \preceq, *\rangle \cong \langle I(L), \preceq, *\rangle$.

Definition 5.15. *For every $i \in \tau$, let $D^1_i = B'^1_i \cap A^1_\tau$, and for every $i \in \tau$ as well, let $D^0_i = B'^0_i \cap (\bigcup_{j\in\tau} A^0_{\tau_j})$.*

Note that both the D^0_i's, and the D^1_i's are members of K and dense in $\Re$. For the D^1_i's it is immediate, as for every $i \in \tau$, $B'^1_i \subseteq A^1_\tau$, so $D^1_i = B'^1_i$. As for every $i \in \tau$, $B_i(0) \subseteq B'^0_i \cap A^0_{\tau_i}$, then as $B_i(0) \subseteq D^0_i$, and $B_i(0)$ is in K, and is dense is $\Re$, D^0_i is also a member of K and dense in $\Re$. Thus, every D^0_i is dense in $\bigcup_{i\in\tau} D^0_i$, and every D^1_i is dense in $\bigcup_{i\in\tau} D^1_i$. Note that, $\bigcup_{i\in\tau} D^0_i = A^1_\tau$, and $\bigcup_{i\in\tau} D^1_i = \bigcup\limits_{i\in\tau} A_{\tau^0_i}$.

Now, as every conjunct of ψ_0 is a conjunct of ψ_1, we obtain from lemma 5.13:

$$K \models \psi_0[B'^0_0, \cdots, B'^0_{n-1}, B'^1_0, \cdots, B'^1_{n_1-1}],$$

so for every $i, j \in \tau$, and $l, k \in \{0, 1\}$, if $i \neq j$, then $B'^l_i \perp B'^k_j$, and if in addition $i \neq j^*$, then $B'^l_i \perp\!\!\!\perp B'^k_j$.

Theorem 5.16. *There is an $\aleph_1$ c.c.c forcing set P, such that*

$$\Vdash_P (\bigcup_{i\in\tau} D^0_i \cong \bigcup_{i\in\tau} D^1_i), \textit{ and}$$

$$\Vdash_P (K \models \psi_0[B'^0_0, \cdots, B'^0_{n-1}, B'^1_0, \cdots, B'^1_{n_1-1}]).$$

Proof. As all the D_i^0's and the D_i^1's are pairwise disjoint and as for every $i \in \tau$, D_i^0 is dense in $\bigcup_{i\in\tau} D_i^0$, and D_i^1 is dense in $\bigcup_{i\in\tau} D_i^1$, the conditions are fit for theorem 2.3. Hence let M be a model which encodes all relevant information, and let P be the forcing set as in theorem 2.3:

$P = \{f : \bigcup_{i\in\tau} D_i^0 \to \bigcup_{i\in\tau} D_i^1 \mid f$ is a finite OP function with respect to $\prec$, G_f^C is cycle free, if $f(a) = b$ then $E^C(a)$, $E^C(b)$ are near, and for every $i \in \tau$, $f(D_i^0) \subseteq D_i^1$, and $f^{-1}(D_i^1) \subseteq D_i^0\}$. $f \leq g$ if $f \subseteq g$.

Then $\Vdash_P (\bigcup_{i\in\tau} D_i^0 \cong \bigcup_{i\in\tau} D_i^1)$. Let G' be a P-generic filter, and let $W = V[G']$.

Suppose that in W, $K\neg \models \psi_0[{B'}_0^0, \cdots, {B'}_{n-1}^0, {B'}_0^1, \cdots, {B'}_{n_1-1}^1]$.

Let B_i^t, B_j^s where $s, t \in \{0,1\}$, $i < n_s$, and $j < n_t$, be such that $B_i^t \wedge B_j^s = 0$ is a conjunct of ψ_0, but B_i^t, B_j^s are not far. Denote $G = B_i^t$, $H = B_j^s$.

This proof will be very similar to the farness proofs of lemma 2.4, and it is recommended to be familiar with it, and with the discussion follows in subsection 2.2; therefore we will omit the details that are already clear, and focus on the more subtle part. In short, we let τ be a P-name such that $\Vdash_P$"τ is an uncountable OP function and $\tau \subseteq G \times H$". We let $\{\langle f_\alpha, \langle a_\alpha, b_\alpha\rangle\rangle \mid \alpha < \aleph_1\}$ be such that for every α, $f_\alpha \Vdash_P \langle a_\alpha, b_\alpha\rangle \in \tau$, and if $\alpha \neq \beta$ then $\langle a_\alpha, b_\alpha\rangle \neq \langle a_\beta, b_\beta\rangle$.

We will reach a contradiction if we find α and β such that $f_\alpha \cup f_\beta \in P$, but $\{\langle a_\alpha, b_\alpha\rangle, \langle a_\beta, b_\beta\rangle\}$ is not OP. We uniformize $\{\langle f_\alpha, \langle a_\alpha, b_\alpha\rangle\rangle \mid \alpha < \aleph_1\}$ as in 2.1, hence we denote $f_\alpha = \{\langle a(\alpha,0), a(\alpha,1)\rangle, \cdots, \langle a(\alpha,2n-2), a(\alpha,2n-1)\rangle\}$. We can assume that all the $a(\alpha,i)$'s are distinct. We also denote $a_\alpha = a(\alpha,2n)$ and $b_\alpha = a(\alpha,2n+1)$. Note that for every $i < n$ there is $j \in \tau$ such that $\langle a(\alpha,2i), a(\alpha,2i+1)\rangle \in D_j^{\epsilon_j} \times D_j^{1-\epsilon_j}$ where $\epsilon_j \in \{0,1\}$.

Let $a(\alpha) = \langle a(\alpha,0), \cdots, a(\alpha,2n+1)\rangle$, $F_1 = \{a(\alpha) \mid \alpha < \aleph_1\}$ and let F be the closure of F_1 in $(\langle \aleph_1, \prec\rangle)^{2n+2}$. We define D, γ_0, $a(i)$, a, W, etc, as in 2.1.

Suppose that we are in the duplication argument. Case 1, and case 2.1 are done exactly as in lemma 2.4. We are now in case 2.2.

Recall that $\langle a(2n), a(2n+1)\rangle \in G \times H$, $a(2n) \in E_{v_0}, a(2n+1) \in E_{v_r}$ where $v_0 \neq v_r$, and $E_{v_0}, \cdots E_{v_r}$ are connected in G_f by a unique path. For every $i < r$, let $\langle a(\eta_i), a(\mu_{i+1})\rangle$ denote the edge in G_f which connects E_{v_i} with $E_{v_{i+1}}$.

Also recall that for every $i \in \tau$, $D_i^0 \subseteq B_i^0$, and $D_i^1 \subseteq B_i^1$. Hence for every $i, j \in \tau$ where $\epsilon_i, \epsilon_j \in \{0,1\}$, if $i \neq j$ then $D_i^{\epsilon_i} \perp D_j^{\epsilon_j}$, and if $i \neq j^*$ as well, then $D_i^{\epsilon_i} \perp\!\!\!\perp D_j^{\epsilon_j}$. There are now several cases to take care of:

Case 2.2.1: Suppose, without loss of generality, that $a(2n) \in G$, $a(\eta_0) \in D_i^{\epsilon_i}$, and $G \perp\!\!\!\perp D_i^{\epsilon_i}$.

As $2n, \eta_0 \in E_{v_0}$, it is possible to construct, in the creation of φ_{v_0} a set of intervals $\{U_t^{P,l} \mid l = 0,1, t \in R_{v_0}\}$ which is $\{2n, \eta_0\}$-OP, and a set of intervals $\{U_t^{R,l} \mid l = 0,1, t \in R_{v_0}\}$ which is $\{2n, \eta_0\}$-OR.

Hence by theorem 2.8 part 2, it is possible to find two conditions f_β, f_γ such that $f_\beta \cup f_\gamma \in P$, but $\{\langle a_\beta, b_\beta\rangle, \langle a_\gamma, b_\gamma\rangle\}$ is OR, a contradiction.

The case where $a(2n+1) \in H$, $a(\mu_r) \in D_i^{\epsilon_i}$ and $H \perp\!\!\!\perp D_i^{\epsilon_i}$, is identical.

Case 2.2.2: There is $i < r$, and $j, k \in \tau$ where $a(\eta_i) \in D_j^{\epsilon_j}$, and $a(\mu_i) \in D_k^{\epsilon_k}$, and $D_j^{\epsilon_j} \perp\!\!\!\perp D_k^{\epsilon_k}$.

Again, as $\eta_i, \mu_i \in E_{v_i}$, it is possible to construct, in the creation of φ_{v_i} a set of intervals $\{U_t^{P,l} \mid l = 0,1, t \in R_{v_i}\}$ which is $\{\eta_i, \mu_i\}$-OP, and a set of intervals $\{U_t^{R,l} \mid l = 0,1, t \in R_{v_i}\}$ which is $\{\eta_i, \mu_i\}$-OR; again, by theorem 2.8 part 2, we reach a contradiction.

Case 2.2.3: Suppose neither of the previous cases happen.

We look at the edges $\{\langle a(\eta_i), a(\mu_{i+1})\rangle \mid i < r\}$ that belong to the unique path which connects E_{v_0}, and E_{v_r}. As $\{a(0), \cdots, a(2n+1)\}$ is a member of F_1, then for every $i < r$, there is $j_i \in \tau$, such that the edge $\langle a(\eta_i), a(\mu_i + 1)\rangle$ is in $D_{j_i}^{\epsilon_{j_i}} \times D_{j_i}^{1-\epsilon_{j_i}}$, and for every $0 < i < r$, there are $j_i, k_i \in \tau$ such that $\langle a(\eta_i), a(\mu_i)\rangle \in D_{j_i}^{\epsilon_{j_i}} \times D_{k_i}^{\epsilon_{k_i}}$. As case 2.2.2 does not happen, it follows that there is $j \in \tau$ such that for every $0 < i < r$, without loss of generality, $j_i = j$, and either $k_i = j$, or $k_i = j^*$ (we are not concerned whether $a(\eta_i)$, or $a(\mu_i)$ are in $D_j^{\epsilon_{j_i}}$).

Note that as $G \perp H$, and case 2.2.1 does not happen, then neither G nor H is very far from $D_j^{\epsilon_j}$ (where $\epsilon_j \in \{0, 1\}$. Hence we can assume that $G = {B'}_j^t$ (where $t \in \{0, 1\}$), and H is either $({B'}_j^s)^*$, or ${B'}_{j^*}^s$, where $s \in \{0, 1\}$. In either case, as $G \perp H$, then $j \neq j^*$.

Claim 5.17. *Let G be a graph, where $v_0, \cdots, v_r$ are vertices in G, and let $T = \langle v_0, v_1\rangle, \cdots, \langle v_r - 1, v_r\rangle$ be a simple trail (i.e. with no cycles) in G which connects between vertices v_0 and v_r, Suppose that we assign to every edge in T one color: either red or blue. Then the edges $\langle v_0, v_1\rangle$, and $\langle v_r - 1, v_r\rangle$ have the same color, if and only if, the number of vertices that have one red edge and one blue edge is even.*

Proof. Obvious. $\square$

We now color the edges of the path which connects E_{v_0}, and E_{v_r}, as follows: For every $i < r$ where the edge $\langle a(\eta_i), a(\mu_i + 1)\rangle$ is in $D_{j_i}^{\epsilon_{j_i}} \times D_{j_i}^{1-\epsilon_{j_i}}$, if $j_i = j$ we color $\langle a(\eta_i), a(\mu_i + 1)\rangle$ in red, and if $j_i = j^*$, we color $\langle a(\eta_i), a(\mu_i + 1)\rangle$ in blue.

Note that, if $\langle a(\eta_i - 1), a(\mu_i)\rangle$, $\langle a(\eta_i), a(\mu_i + 1)\rangle$ have different colors. then $k_i = j^*$, and if the colors are the same then $k_i = j$.

As $j \neq j^*$, then $k_i = j^*$ means that $D_j^{\epsilon_j} \perp D_{k_i}^{\epsilon_{k_i}}$. Hence, in the construction of φ_{v_i}, it is possible to construct a set of intervals $\{U_t^l \mid l = 0, 1, t \in R_{v_i}\}$ which is $\{\eta_i, \mu_i\}$-OR. Same, if $k_i = j$, then $D_j^{\epsilon_j} \perp (D_{k_i}^*)^{\epsilon_{k_i}}$, and in the construction of φ_{v_i}, it is possible to construct a set of intervals $\{U_t^l \mid l = 0, 1, t \in R_{v_i}\}$ which is $\{\eta_i, \mu_i\}$-OP.

We see that for every E-slice E_{v_i} we can provide either a set of intervals which is $\{\eta_i,\mu_i\}$-OP, or a set of intervals which is $\{\eta_i,\mu_i\}$-OR.

We now turn again to look at G, and H, which must be one of the two following options:

case a: Without loss of generality, $G = {B'}^t_j$, $H = ({B'}^s_j)^*$, where $t, s \in \{0,1\}$ (the case where $G = B^t_{j^*}$, $H = (B^s_{j^*})^*$ is identical).

Recall that $a(2n) \in G, a(2n+1) \in H$. Suppose that, without loss of generality, $a(2n) \in E_{v_0}$, and $a(2n+1) \in E_{v_r}$ If $a(\eta_0) \in D_j^{\epsilon_{j0}}$ (hence $\langle a(\eta_0), a(\mu_1)\rangle$ is colored in red), then as $D_j^{\epsilon_{j0}} \subseteq B_j^{\epsilon_{j0}}$, $D_j^{\epsilon_{j0}} \perp (B^t_j)^*$. As $G = B^t_j$ we can construct, in the creation of φ_{v_0}, a set of intervals which is $\{2n, \eta_0\}$-OP. If $a(\eta_0) \in D_{j^*}^{\epsilon_{j_0^*}}$ (hence $\langle a(\eta_0), a(\mu_1)\rangle$ is colored in blue), then as $D_{j^*}^{\epsilon_{j_0^*}} \perp B^t_j$, we can construct, in the creation of φ_{v_0}, a set of intervals which is $\{2n, \eta_0\}$-OR.

If $a(\mu_r) \in D_j^{\epsilon_{jr}}$ (hence $\langle a(\eta_{r-1}), a(\mu_r)\rangle$ is colored in red), then as $D_j^{\epsilon_{jr}} \perp (B^t_j)^*$, we can construct, in the creation of φ_{v_r}, a set of intervals which is $\{2n+1, \mu_r\}$-OR. Last, if $a(\mu_r) \in D_{j^*}^{\epsilon_{j_r^*}}$ (hence $\langle a(\eta_{r-1}), a(\mu_r)\rangle$ is colored in blue), we can construct, in the creation of φ_{v_r}, a set of intervals which is $\{2n+1, \mu_r\}$-OP.

Let $S \in \{1, \cdots, r-1\}$ be such that $S \overset{def}{=} \{0 < i < r \mid$ the set of intervals of φ_i is $\{\eta_i, \mu_i\}$-OR$\}$. Suppose that for every $0 < i < r$, $i \notin S$ the set of intervals of φ_i is $\{\eta_i, \mu_i\}$-OP.

if $\langle a(\eta_0), a(\mu_1)\rangle$, and $\langle a(\eta_r - 1), a(\mu_r)\rangle$ are of the same color, then either the set of intervals of φ_{v_0} can be constructed to be $\{2n, \eta_0\}$-OP, and the set of intervals of φ_{v_r} can be constructed to be $\{\mu_r, 2n+1\}$-OR, or the other way around.

From claim 5.17 we can construct the sets of intervals in G_f such that $|S|$ is even. As, without loss of generality, we constructed the set of intervals of φ_{v_0} to be $\{2n, \eta_0\}$-OP, and the set of intervals of φ_{v_r} to be $\{\mu_r, 2n+1\}$-OR, by theorem 2.8 part 1, it is possible to find two conditions f_β, f_γ such that $f_\beta \cup f_\gamma \in P$, but $\{\langle a_\beta, b_\beta\rangle, \langle a_\gamma, b_\gamma\rangle\}$ is OR, a contradiction.

Suppose that $\langle a(\eta_0), a(\mu_1)\rangle$, and $\langle a(\eta_r - 1), a(\mu_r)\rangle$ are of different colors, then it can be constructed that both the intervals of φ_{v_0} and φ_{v_r} will have the OP property, or both have the OR-property. Without loss of generality we constructed the set of intervals of φ_{v_0} to be $\{2n, \eta_0\}$-OP, and the set of intervals of φ_{v_r} to be $\{\mu_r, 2n+1\}$-OP as well, and again, by theorem 2.8 part 1, it is possible to find two conditions f_β, f_γ such that $f_\beta \cup f_\gamma \in P$, but $\{\langle a_\beta, b_\beta\rangle, \langle a_\gamma, b_\gamma\rangle\}$ is OR, a contradiction.

case b: Without loss of generality, $G = {B'}^t_j$, $H = ({B'}^s_{j^*})$, where $t, s \in \{0, 1\}$.

Suppose, without loss of generality, that $a(2n) \in E^v_0, a(2n+1) \in E^v_r$. As in previous case, if $a(\eta_0) \in D_j^{\epsilon_{j_0}}$, hence $\langle a(\eta_0), a(\mu_1)\rangle$ is colored in red, we can construct in the creation of φ_{v_0}, a set of intervals which is $\{2n, \eta_0\}$-OP, and if $a(\eta_0) \in D_{j^*}^{\epsilon_{j_0^*}}$, hence $\langle a(\eta_0), a(\mu_1)\rangle$ is colored in blue, we can construct in the creation of φ_{v_0}, a set of intervals which is $\{2n, \eta_0\}$-OR. If $a(\mu_r) \in D_j^{\epsilon_{j_r}}$ (hence $\langle a(\eta_{r-1}), a(\mu_r)\rangle$ is colored in red), then as $D_j^{\epsilon_{j_r}} \perp B^t_{j^*}$, we can construct in the creation of φ_{v_r}, a set of intervals which is $\{2n+1, \mu_r\}$-OR. Last, if $a(\mu_r) \in D_{j^*}^{\epsilon_{j_r^*}}$, then as $D_{j^*}^{\epsilon_{j_r^*}} \perp (B^t_{j^*})^*$ (hence $\langle a(\eta_{r-1}), a(\mu_r)\rangle$ is colored in blue), we can construct in the creation of φ_{v_r}, a set of intervals which is $\{2n+1, \mu_r\}$-OP.

If $\langle a(\eta_0), a(\mu_1)\rangle$, and $\langle a(\eta_r - 1), a(\mu_r)\rangle$ are of the same color, then the set of intervals of φ_{v_0} can be constructed to be $\{2n, \eta_0\}$-OP, and the set of intervals of φ_{v_r} can be constructed to be $\{2n+1, \mu_e\}$-OP as well. Let S be as in previous case. Then by claim 5.17, $|S|$ is even, and by theorem 2.8 part 1, it is possible to find two conditions f_β, f_γ such that $f_\beta \cup f_\gamma \in P$, but $\{\langle a_\beta, b_\beta\rangle, \langle a_\gamma, b_\gamma\rangle\}$ is OR, a contradiction.

If $\langle a(\eta_0), a(\mu_1)\rangle$, and $\langle a(\eta_r - 1), a(\mu_r)\rangle$ are of different color, then the set of intervals of φ_{v_0} can be constructed to be $\{2n, \eta_0\}$-OP, and the set of intervals of φ_{v_r} can be constructed to be $\{2n+1, \mu_e\}$-OR. Then by claim 5.17 $|S|$ is odd, and again by theorem 2.8 part 1, we reach a contradiction. That sums the proof of 5.16. $\square$

We come to the end of the proof of lemma 5.1. We define the forcing set acquired from theorem 5.16 as $P_{\nu+1}$. Our last concern is to make sure that $P_{\nu+1}$ meets the requirements of lemma 5.1.

As $\Vdash_{P_{\nu+1}} (K \models \psi_0[{B'}^0_0, \cdots, {B'}^0_{n-1}, {B'}^1_0, \cdots, {B'}^1_{n_1-1}])$, $\Vdash_{P_{\nu+1}} (K \models \varphi_0[{B'}^0_0, \cdots, {B'}^0_{n-1}])$ as well. Then from lemma 3.15, $\Vdash_{P_{\nu+1}} (K \models \varphi_1[{B'}^0_0, \cdots, {B'}^0_{n-1}; \bigcup_{j\in\tau_i} {B'}^0_j | i < n])$.

As $A_{\tau_i} \subseteq \bigcup_{j\in\tau_i} {B'}^0_j$, and $B_0(i) \subseteq {B'}^0_i$, we obtain from lemma 3.16, $\Vdash_Q (K \models \varphi_1[B_0(0), \cdots, B_0(n-1); A_{\tau_i} | i < n])$.

By the definition of the $B_0(i)$'s, for every $i < n$ $B_0(i) \in F_i$, and $B_0(i^*) = B^*_0(i)$; hence for every $i < n$ we define $B'_i = B_0(i)$, then

$\Vdash_{P_{\nu+1}} (K \models \varphi_1[B'_0, \cdots, B'_{n-1}; A_{\tau_i} | i < n])$, As required. That settles lemma 5.1 $\square$

The proof of the main theorem of this work, theorem 3.3, is now complete. We have managed to create a universe in which $\langle K^H, \preceq, * \rangle$ is isomorphic to a given finite distributive lattice with an involution.

One may ask whether it is possible to construct K^H to become isomorphic to any given countable distributive lattice. In [ARS] section 11, a guideline for well-founded lattices was given; other sort of lattices still form a challenge.

In this work we have seen several techniques for solving questions which concern quasi ordered uncountable sets in $\Re$. Many other results, using the same kind of tools such as the club method, or the explicit contradiction method, can be found in [ARS].

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