

Comp 311  
Principles of Programming Languages  
Lecture 7  
Meta-interpreters

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# Denotational Semantics

- The primary alternative to *syntactic* semantics is *denotational* semantics. A denotational semantics maps abstract syntax trees to a set of *denotations* (mathematical values like numbers, lists, and functions).
- Simple denotations like numbers and lists are essentially the same mathematical object as syntactic values: they have simple inductive definitions with exactly the same structure as the corresponding abstract syntax trees.
- But denotations can also be complex mathematical objects like *functions* or *sets*. For example, the denotation for a lambda-expression in “pure” (functional) Scheme is a function mapping denotations to denotations--*not* some syntax tree as in a syntactic semantics.

# Meta-interpreters

- Denotational semantics is rooted in mathematical logic: the semantics of terms (expressions) in the predicate calculus is defined denotationally by *recursion* on the syntactic structure of terms. The meaning of each term is a value in an *algebraic structure*.
- In the realm of programming languages, purely functional interpreters (defined by pure recursion on the structure of ASTs) constitute a restricted form of denotational definition.
  - The defect is that the output of the interpreter is restricted to values that can be characterized syntactically. (How do you output a function?)
  - On the other hand, interpreters naturally introduce a simple form of functional abstraction. A recursive interpreter accepts an extra input, an environment mapping free variables to values, thus defining the meaning of a program expression as a function from environments to values.
  - Syntactic interpreters are *not denotational* because they transform ASTs. A denotational interpreter uses pure structural recursion. To handle the bindings to variables, it cannot perform substitutions; it must maintain an environment of bindings instead.

# Meta-interpreters cont.

- Interpreters written in a denotational style are often called *meta*-interpreters because they are defined in a meta-mathematical framework where programming language expressions are objects in the framework. The definition of the interpreter is a level above definitions of functions in the language being defined.
- In mathematical logic, meta-level definitions are expressed informally as definitions of mathematical functions.
- In program semantics, meta-level definitions are expressed in a convenient functional framework with a semantics that is easily defined and understood using informal mathematics. *Formal* denotational definitions are written in a mathematical meta-language corresponding to some formulation of a *Universal Domain* (a mathematical domain in which all relevant programming language domains can be simply embedded, usually as projections). This material is subject of a graduate level course on domain theory.
- A functional interpreter for language L written in a functional subset of L is called a *meta-circular* interpreter. It really isn't circular because it reduces the meaning of all programs to a single purely functional program which can be understood independently using simple mathematical machinery (inductive definitions over familiar mathematical domains).

# Denotational Building Blocks

- Inductively defined ASTs for program syntax. We have thoroughly discussed this topic.
- What about denotations? For now, we will only use simple inductively defined values (without functional abstraction) like numbers, lists, tuples, etc.
- What about environments? Mathematicians like to use functions. An environment is a function from variables to denotations. But environment functions are special because they are *finite*. Software engineers prefer to represent them as lists of pairs binding variables to denotations.
- In “higher-order” languages, functions are data objects. How do we represent them? For now we will use ASTs possibly supplemented by simple denotations (as described above).

# Critique of Deferred Substitution Interpreter from Lecture 6

- How did we represent the denotations of lambda-expressions (functions)? By their ASTs. Is this implementation correct? No!

- Counterexample: twice

```
(let  
  ([twice (lambda (f) (lambda (x) (f (f x))))]  
   [x 5])  
  ((twice (lambda (y) (+ x y))) 0))
```

# Evaluate (syntactically)

```
(let  
  [(twice (lambda (f) (lambda (x) (f (f x))))]  
   (x 5)]  
  ((twice (lambda (y) (+ x y))) 0))
```

=>

```
((lambda (f) (lambda (x) (f (f x))))  
 (lambda (y) (+ 5 y))) 0)
```

=>

```
(lambda (x)  
  ((lambda (y) (+ 5 y)) ((lambda (y) (+ 5 y)) x)))  
  0)
```

=>

```
((lambda (y) (+ 5 y)) ((lambda (y) (+ 5 y)) 0)) =>  
(lambda (y) (+ 5 y)) (+ 5 0) =>  
(lambda (y) (+ 5 y)) 5 => (+ 5 5) => 10
```

# Closures Are Essential

- **Exercise:** evaluate the same expression using our broken interpreter.
- The computed “answer” is 0!
- The interpreter uses the wrong binding for the free variable  $x$  in `(lambda (y) (+ x y))`.
- The environment records deferred substitutions. When we pass a function as an argument, we need to pass a “package” including the deferred substitutions. Why? The function will be applied in a *different* environment which may associate the *wrong* bindings with free variables. In the PL (programming languages) literature, these packages (code representation, environment) are called *closures*.
- Note the similarity between this mistake and the “capture of bound variables.”
- Unfortunately, this mistake has been labeled as a feature rather than a bug in much of the PL (programming languages) literature. It is called “dynamic scoping” rather than a horrendous mistake.

# Correct Semantic Interpretation

```
(define-struct (closure proc env))
;; V = Const | Closure ; revises our former definition of V
;; Binding = (make-Binding Sym V) ; Note: Sym not Var
;; Env = (listOf Binding)
;; Closure = (make-closure Proc Env)
;; R Env → V
(define eval
  (lambda (M env)
    (cond
      ((var? M) (lookup (var-name M) env))
      ((const? M) M)
      ((proc? M) (make-closure M env))
      ((add? M) ; ; M has form (+ l r); (add-left M) is l
        (add (eval (add-left M) env) (eval (add-right M) env)))
      (else ; ; M has form (N1 N2) ; N1 should eval to a Closure
        (apply (eval (app-rator M) env) (eval (app-rand M) env))))))
;; Closure V → V
(define apply
  (lambda (cl v)
    (eval (proc-body (closure-proc cl))
          (cons (make-binding (proc-param (closure-proc cl)) v)
                (closure-env cl))))
```