Major Challenge

- LC does not include a recursive binding operation (like Scheme `letrec` or `local`). How would we define `eval` for such a construct?
- Key problem: the closure structure for a recursive `lambda` must include an environment that refers to itself!
- In imperative Java, how would we construct such an environment. Hint: how do we build “circular” data structures in general in Java? Imperativity is brute force. But it works. We will use it in Project 3 and thereafter.
Minor Challenge

• How could we define an environment that refers to itself in *functional* Scheme (or Ocaml)?

• Key problem: observe that in `let` and `lambda` the expression defining the value of a variable cannot refer to itself.

• Solution: does functional Scheme (or Ocaml) contain a recursive binding construct?

• What environment representation must we use?
Advantages of Representing Environments as Functions

Languages that support functions as values (or an OO equivalent like anonymous inner
classes [Java] or anonymous delegates [C#]) support the dynamic definition of recursive
functions. So we can write a purely functional interpreter that assigns a meaning to recursive
binding by constructing a new environment (a function) that recurs on itself (refers to itself).
In Scheme, given a function $e$ that represents the current environment, we can extend $e$
with a new binding of symbol $f$ to an AST $\text{rhs}$ (right-hand-side) that is evaluated in the extended
environment by constructing the environment

$$(\text{define new-e (lambda (sym) (if (=? sym f) (eval rhs new-e) (e sym))))}$$

where $\text{eval}$ is the meta-interpreter. We can introduce recursive binding without the
side effect introduced by $\text{define}$ by using the the Scheme construct $\text{letrec}$.

Scheme $\text{letrec}$ is akin to $\text{let}$ except that it performs recursive binding instead of conventional
binding, $i.e.$, that the new environment created by $\text{letrec}$ is used to evaluate all
subexpressions on the rhs of the symbol definition added by $\text{letrec}$ (see the syntax for $\text{let}$
in the previous lecture). Note that the binding of the new symbol is unavailable (sometimes
represented by the error value $\text{void}$) until the evaluation of the rhs is complete. This trick
works for $\text{letrec}$ constructs that introduce new function definitions but not for other kinds of
data unless the constructors for that form of data are “lazy” (delaying the evaluation of their
arguments until demanded by an accessor operation).
A Bigger Challenge

- Assume that we want to write LC in a purely functional language without a recursive binding construct (say functional Scheme without \texttt{letrec}).

- Key problem: must expand \texttt{letrec} into \texttt{lambda}.

- No simple solution to this problem. We need to invoke syntactic magic or (equivalently) develop some sophisticated mathematical machinery.
Key Intuitions

• Computation is incremental—not monolithic.
• Slogan: general computation is successive approximation (typically in response to successive demands for more information).
• Familiar example: a program mapping a potentially infinite input stream of characters to a potentially infinite output stream of characters.
• Generalization: infinite trees mapped to infinite trees.
A (Scott) domain of computations $D$ (like streams, trees, partial functions as graphs) is a partially ordered set (po) with the following properties:

- $D$ has a countable subset $B$ (set of finite approximations), called the finitary basis, which is a po that is finitely consistent, i.e., closed under LUBs on bounded subsets (implies a unique least elt $\bot$ exists). We will restrict our attention to finitary bases $B$ where no element $b$ in $B$ is the LUB of an infinite subset of $B$.

- $D$ is chain-complete: every chain $b_0 \leq b_1 \leq \ldots \leq b_k \leq \ldots$ (a countable ascending sequence) in $B$ has a lub in $D$.

- A po with that is chain-complete is called a cpo (complete partial order).

Note: in the reference monograph, directed sets are used instead of chains. When the finitary basis is countable, it is straightforward to show that chain-complete and directed-complete are equivalent.

Examples:

- flat domains (integers, booleans, finite trees with no undefined ($\bot$) leaves)
- lazy tree domains (potentially infinite trees with a finite set of node types and undefined ($\bot$) leaves)
Key Mathematical Concepts

Computable functions on domains:

• monotonic (universal)
• continuous (universal)
• strict (typical)

For a brief, intuitive overview, see the topic notes for lecture 11

https://www.cs.rice.edu/~javaplt/411/19-spring/Notes/11/06.html

For an in-depth treatment of (Scott) domains, see the monograph linked under references for lecture 10.
Examples

Domains

• flat domains
• strict function spaces on flat domains (CBV)
• lazy trees of boolean (of D where D is flat)
• continuous functions $A \rightarrow B$ where $A$ and $B$ are domains

The notion of continuity here is very important; it enables interchanging function application and the LUB operation on chains.