Two-Factor Full Factorial Design <u>with</u> Replications

Dr. John Mellor-Crummey

Department of Computer Science
Rice University

johnmc@cs.rice.edu

Goals for Today

Understand

- **Two-factor Full Factorial Design with Replications**
  - —motivation & model
  - —model properties
  - —estimating model parameters
  - —estimating experimental errors
  - —allocating variation to factors and interactions
  - —analyzing significance of factors and interactions
  - —confidence intervals for effects
  - —confidence intervals for interactions

Two-factor Design with Replications

- **Motivation**
  - —two-factor full factorial design without replications
    - – helps estimate the effect of each of two factors varied
    - – assumes negligible interaction between factors
      - effects of interactions are ignored as errors
  - —two-factor full factorial design with replications
    - – enables separation of experimental errors from interactions
- **Example: compare several processors using several workloads**
  - —factor A: processor type   factor B: workload type
  - —no limits on number of levels each factor can take
  - —full factorial design → study all workloads x processors
- **Model:** $y_{ijk} = \mu + \alpha_j + \beta_i + \gamma_{ij} + e_{ijk}$
  - —y_{ijk} - response with factor A at level j and factor B at level i
  - —μ - mean response
  - —α_j - effect of level j for factor A
  - —β_i - effect of level i for factor B
  - —γ_{ij} - effect of interaction of factor A at level j with B at level i
  - —e_{ijk} - error term

Model Properties

- **Effects**
  - —sums are 0

$$\sum_j \alpha_j = 0, \ \sum_i \beta_i = 0$$

- **Interactions**
  - —row sums are 0

$$\sum_{j=1}^{a} \gamma_{1j} = \sum_{j=1}^{a} \gamma_{2j} = \ \ldots \ = \sum_{j=1}^{a} \gamma_{bj} = 0$$

  - —column sums are 0

$$\sum_{i=1}^{b} \gamma_{i1} = \sum_{i=1}^{b} \gamma_{i2} = \ \ldots \ = \sum_{i=1}^{b} \gamma_{ia} = 0$$

- **Errors**
  - —errors in each experiment sum to 0

$$\sum_{k=1}^{r} e_{ijk} = 0, \quad \forall i, j$$

Estimating Model Parameters I

- Organize measured data for two-factor full factorial design as
  - — b x a matrix of cells: (i,j) = factor B at level i and factor A at level j

    columns = levels of factor A rows = levels of factor B
  - —each cell contains r replications
- Begin by computing averages
  - —observations in each cell

$$\bar{y}_{ij.} = \mu + \alpha_j + \beta_i + \gamma_{ij}$$ (∑ errors = 0)

  - —each row

$$\bar{y}_{i..} = \mu + \beta_i$$ (∑ column effects, errors, and interactions = 0)

  - —each column

$$\bar{y}_{.j.} = \mu + \alpha_j$$ (∑ row effects, errors, and interactions = 0)

  - —overall

$$\bar{y}_{...} = \mu$$ (∑ row & column effects, errors, and interactions = 0)

Estimating Model Parameters II

- **Estimate effects from equations for averages**
  - **mean**

$$\mu = \bar{y}_{...}$$

  - **effect of factor A at level j**

$$\bar{y}_{.j.} = \mu + \alpha_j \quad \longrightarrow \quad \alpha_j = \bar{y}_{.j.} - \bar{y}_{...}$$

  - **effect of factor B at level i**

$$\bar{y}_{i..} = \mu + \beta_i \quad \longrightarrow \quad \beta_i = \bar{y}_{i..} - \bar{y}_{...}$$

  - **overall**

$$\bar{y}_{ij.} = \mu + \alpha_j + \beta_i + \gamma_{ij} \quad \longrightarrow \quad \gamma_{ij} = \bar{y}_{ij.} - \bar{y}_{.j.} - \bar{y}_{i..} + \bar{y}_{...}$$

- **How to compute model parameter estimates?**
  - **use table as for two-factor full factorial design without replications, <u>except</u> use cell means to compute row and column effects**

Example: Workload Code Size vs. Processor

- **Code size for 5 different workloads on 4 different processors**
- **Codes written by 3 programmers with similar backgrounds**

Workload	W	X	Y	Z
I	7006	12042	29061	9903
	6593	11794	27045	9206
	7302	13074	30057	10035
J	3207	5123	8960	4153
	2883	5632	8064	4257
	3523	4608	9677	4065
K	4707	9407	19740	7089
	4935	8933	19345	6982
	4465	9964	21122	6678
L	5107	5613	22340	5356
	5508	5947	23102	5734
	4743	5161	21446	4965
M	6807	12243	28560	9803
	6392	11995	26846	9306
	7208	12974	30559	10233

- **What kind of model is needed: additive or multiplicative?**

Example: Workload Code Size vs. Processor

Transformed data ($\log_{10}$)

Workload	W	X	Y	Z
I	3.8455	4.0807	4.4633	3.9958
	3.8191	4.0717	4.4321	3.9641
	3.8634	4.1164	4.4779	4.0015
J	3.5061	3.7095	3.9523	3.6184
	3.4598	3.7507	3.9066	3.6291
	3.5469	3.6635	3.9857	3.6091
K	3.6727	3.9735	4.2953	3.8506
	3.6933	3.9510	4.2866	3.8440
	3.6498	3.9984	4.3247	3.8246
L	3.7082	3.7492	4.3491	3.7288
	3.7410	3.7743	4.3636	3.7585
	3.6761	3.7127	4.3313	3.6959
M	3.8330	4.0879	4.4558	3.9914
	3.8056	4.0790	4.4289	3.9688
	3.8578	4.1131	4.4851	4.0100

Example: Workload Code Size vs. Processor

Computing effects with replications

Workload	W	X	Y	Z	row sum	row mean	row effect
I	3.8427	4.0896	4.4578	3.9871	16.3772	4.0943	0.1520
J	3.5043	3.7079	3.9482	3.6188	14.7792	3.6948	-0.2475
K	3.6720	3.9743	4.3022	3.8397	15.7882	3.9470	0.0047
L	3.7084	3.7454	4.3480	3.7277	15.5296	3.8824	-0.0599
M	3.8321	4.0933	4.4566	3.9900	16.3721	4.0930	0.1507
column sum	18.5594	19.6105	21.5128	19.1635	78.8462		
column mean	3.7119	3.9221	4.3026	3.8327		3.9423	
column effect	-0.2304	-0.0202	0.3603	-0.1096			

- **Each entry in pale yellow is the average of cell observations**
- **Row and column effects computed as before from cell avgs**
- **Interpretation**
  - —code size avg workload on avg processor = 8756 ($10^{3.9423}$)
  - —processor W requires .2304 less log code size (factor of 1.70)
  - —processor Y requires .3603 more log code size (factor of 2.29)
  - —difference between avg code size for W and Y is factor of 3.90

Example: Workload Code Size vs. Processor

Workload	W	X	Y	Z	row sum	row mean	row effect
I	3.8427	4.0896	4.4578	3.9871	16.3772	4.0943	0.1520
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column effect	-0.2304	-0.0202	0.3603	-0.1096			

Compute interactions as $\gamma_{ij} = \bar{y}_{ij.} - (\mu + \alpha_j + \beta_i)$

Workload	W	X	Y	Z
I	-0.0212	0.0155	0.0032	0.0024
J	0.0399	0.0333	-0.1069	0.0337
K	-0.0447	0.0475	-0.0051	0.0023
L	0.0564	-0.1168	0.1054	-0.0450
M	-0.0305	0.0205	0.0033	0.0066

- **Interaction constraints: row and column sums are 0**
- **Interpretation**
  - —interaction of I & W requires factor of $10^{.0212}$ < code than avg workload on W
  - —interaction of I & W requires factor of $10^{.0212}$ < code than I on avg processor

$\boxed{10^{.0212} = 1.05}$

Estimating Experimental Errors

- **Predicted response of (i,j)th experiment**

$$\hat{y}_{ij} = \mu + \alpha_j + \beta_i + \gamma_{ij} = \bar{y}_{ij.}$$

- **Prediction error = observation - cell mean**

$$e_{ijk} = y_{ijk} - \hat{y}_{ij} = y_{ijk} - \bar{y}_{ij.}$$

Allocating Variation

- **Total variation of y can be allocated to**
  - —the two factors
  - —their interactions
  - —errors
- **Square both sides of model equation**

$$\sum_{i,j,k} y_{ijk}^2 = abr\mu^2 + br\sum_j \alpha_j^2 + ar\sum_i \beta_i^2 + r\sum_{i,j} \gamma_{ij}^2 + \sum_{i,j,k} e_{ijk}^2 + \boxed{\text{cross product terms (all = 0)}}$$

SSY = SS0 + SSA + SSB + SSAB + SSE

- **Total variation**

  SST = SSY - SS0 = SSA + SSB + SSAB + SSE
- **Compute SSE (easiest from other sums of squares)**

  SSE = SSY - SS0 - SSA - SSB - SSAB
- **Percent variation for each factor, interaction and errors**
  - —factor A: 100(SSA/SST) factor B: 100(SSB/SST)
  - —interactions: 100(SSAB/SST) error: 100(SSE/SST)

Allocating Variation for Code Size Study

Workload	W	X	Y	Z	row effect
I	3.8455	4.0807	4.4633	3.9958	0.1520
	3.8191	4.0717	4.4321	3.9641	
	3.8634	4.1164	4.4779	4.0015	
J	3.5061	3.7095	3.9523	3.6184	-0.2475
	3.4598	3.7507	3.9066	3.6291	
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column effect	-0.2304	-0.0202	0.3603	-0.1096	3.9423

Workload	W	X	Y	Z
I	-0.0212	0.0155	0.0032	0.0024
J	0.0399	0.0333	-0.1069	0.0337
K	-0.0447	0.0475	-0.0051	0.0023
L	0.0564	-0.1168	0.1054	-0.0450
M	-0.0305	0.0205	0.0033	0.0066

% interaction = 100*SSAB/SST
= 100*.1548/4.44 = 3.48%

$$SSY = \sum_{i,j,k} y_{ijk}^2 = (3.8455)^2 + (3.8191)^2 + \ldots + (4.0100)^2 = 936.95$$

$$SS0 = abr\mu^2 = (5)(4)(3)(3.9423)^2 = 932.51 \quad SST = SSY - SS0 = 936.95 - 932.51 = 4.44$$

$$SSA = br\sum_j \alpha_j^2 = (5)(3)((-.2304)^2 + (-.0202)^2 + (.3603)^2 + (-.1096)^2) = 2.93$$

$$SSB = ar\sum_j \beta_j^2 = (4)(3)((.1520)^2 + (-.2475)^2 + (.0047)^2 + (-.0599)^2 + (.1507)^2) = 1.33$$

$$SSAB = r\sum_{i,j} \gamma_{ij}^2 = (3)((-.0212)^2 + (.0399)^2 + \ldots + (.0066)^2) = .1548$$

% workload = 100*SSA/SST
= 100*2.93/4.44 = 66.0%

% procs = 100*SSB/SST
= 100*1.33/4.44 = 29.9%

Degrees of Freedom

- **Degrees of freedom**

$$\begin{array}{ccccccccccccc} SSY & = & SS0 & + & SSA & + & SSB & + & +SSAB & + & SSE \\ abr & = & 1 & + & (a-1) & + & (b-1) & + & (a-1)(b-1) & + & ab(r-1) \end{array}$$

sum of abr obs.
all independent

single term μ^2:
repeated abr times

sum of $(\alpha_j)^2$:
a-1 independent
$(\sum\alpha_j = 0)$

sum of $(\beta_i)^2$:
b-1 independent
$(\sum\beta_i = 0)$

sum of $(\gamma_{ij})^2$,
(a-1)(b-1)
independent
0 across row,
down column

sum of $(e_{ijk})^2$:
ab(r-1)
independent

Analyzing Significance

- **Divide each sum of squares by its DOF to get mean squares**
  - —**MSA = SSA/(a-1)**
  - —**MSB = SSB/(b-1)**
  - —**MSAB = SSAB/((a-1)(b-1))**
  - —**MSE = SSE/(ab(r-1))**

$$s_e = \sqrt{MSE}$$

- **Are effects of factors and interactions significant at α level?**
  - —**Test MSA/MSE against F-distribution $F_{[1-\alpha;\ a-1;\ ab(r-1)]}$**
  - —**Test MSB/MSE against F-distribution $F_{[1-\alpha;\ b-1;\ ab(r-1)]}$**
  - —**Test MSAB/MSE against F-distribution $F_{[1-\alpha;\ (a-1)(b-1);\ ab(r-1)]}$**

	Sum of Squares	% var	DOF	Mean Square	F computed	F table
processors	2.929504	65.96	3	0.9765	1340.0121	2.23
workloads	1.328183	29.9	4	0.332	455.65256	2.09
interactions	0.154789	3.48	12	0.0129	17.700875	1.71
errors	0.029149	0.66	40	0.0007		

for α=.1

Variance for μ

- **Expression for μ in terms of random variables y_{ij}'s**

$$\mu = \frac{1}{abr}\sum_{i=1}^{b}\sum_{j=1}^{a}\sum_{k=1}^{r} y_{ijk}$$

$$\mathrm{Var}(\mu) = Var\left(\frac{1}{abr}\sum_{i=1}^{b}\sum_{j=1}^{a}\sum_{k=1}^{r} y_{ijk}\right)$$

Assuming
errors normally distributed
zero mean, variance $(\sigma_e)^2$
What is the variance for μ?

$$\sigma_\mu^2 = \left(abr\left(\frac{1}{abr}\right)^2\right)\sigma_e^2$$

$$= \frac{1}{abr}\sigma_e^2$$

Variance for α_j

- **Expression for α_j in terms of random variables y_{ij}'s**

$$\alpha_j = \bar{y}_{.j.} - \bar{y}_{...} = \frac{1}{br}\sum_{i=1}^{b}\sum_{k=1}^{r} y_{ijk} - \frac{1}{abr}\sum_{i=1}^{b}\sum_{l=1}^{a}\sum_{k=1}^{r} y_{ilk}$$

- **What is the coefficient a_{ikj} for each y_{ilk} for α_j?**

$$a_{ilk} = \begin{cases} \dfrac{1}{br} - \dfrac{1}{abr} & l = j \\ -\dfrac{1}{abr} & \text{otherwise} \end{cases}$$

Assuming
errors normally distributed
zero mean, variance $(\sigma_e)^2$
What is the variance for α_j?

$$\boxed{\begin{aligned} \sigma_{\alpha_j}^2 &= \left(br\left(\frac{1}{br} - \frac{1}{abr}\right)^2 + (abr - br)\left(\frac{1}{abr}\right)^2 \right)\sigma_e^2 \\ &= \frac{(a-1)}{abr}\sigma_e^2 \end{aligned}}$$

Parameter Estimation for Two Factors w/ Repl

Parameter	Estimate	Variance
μ	$\bar{y}_{...}$	s_e^2/abr
α_j	$\bar{y}_{.j.}-\bar{y}_{...}$	$s_e^2(a-1)/abr$
β_i	$\bar{y}_{i..}-\bar{y}_{...}$	$s_e^2(b-1)/abr$
γ_{ij}	$\bar{y}_{ij.}-\bar{y}_{i..}-\bar{y}_{.j.}+\bar{y}_{...}$	$s_e^2(a-1)(b-1)/abr$
$\sum_{j=1}^{a} h_j\alpha_j, \sum_{j=1}^{a} h_j=0$	$\sum_{j=1}^{a} h_j\bar{y}_{.j.}$	$s_e^2\sum_{j=1}^{a} h_j^2/br$
$\sum_{i=1}^{b} h_i\beta_i, \sum_{i=1}^{b} h_i=0$	$\sum_{i=1}^{b} h_i\bar{y}_{i..}$	$s_e^2\sum_{i=1}^{b} h_i^2/ar$
s_e^2	$\left(\sum e_{ijk}^2\right)/(ab(r-1))$	

DOF of errors = ab(r-1)

Confidence Intervals for Effects

- **Variance of processor effects**

$$s_{\alpha_j} = s_e\sqrt{\frac{(a-1)}{abr}} = .0267\sqrt{\frac{4-1}{4\times5\times3}} = .0060$$

- **Error degrees of freedom = ab(r-1) = 5 x 4 x 2 = 40**
  - **—can use unit normal variate rather than t-variate since DOF large**

- **For 90% confidence level: $z_{.95}$ = 1.645**

$$\alpha_j \pm z_{.95}s_{\alpha_j}$$

for processor W

—example:

$$\alpha_1 \pm z_{.95}s_{\alpha_1} = -.2304 \pm (1.645)(.0060) = (-.2406, -.2203)$$

does not include 0: effect is significant

Confidence Intervals for Effects

parameter	mean effect	std deviation	confidence interval
Mu	3.9423	0.0035	(3.9366 , 3.9480)
processors			
W	-0.2304	0.0070	(-0.2419 , -0.2190)
X	-0.0202	0.0070	(-0.0317 , -0.0087)
Y	0.3603	0.0070	(0.3488 , 0.3717)
Z	-0.1096	0.0070	(-0.1211 , -0.0982)
workloads			
I	0.1520	0.0060	(0.1420 , 0.1619)
J	-0.2475	0.0060	(-0.2574 , -0.2376)
K	0.0047	0.0060	(-0.0052 , 0.0147)
L	-0.0599	0.0060	(-0.0698 , -0.0500)
M	0.1507	0.0060	(0.1408 , 0.1606)

pink cells are not significant

Confidence Intervals for Interactions

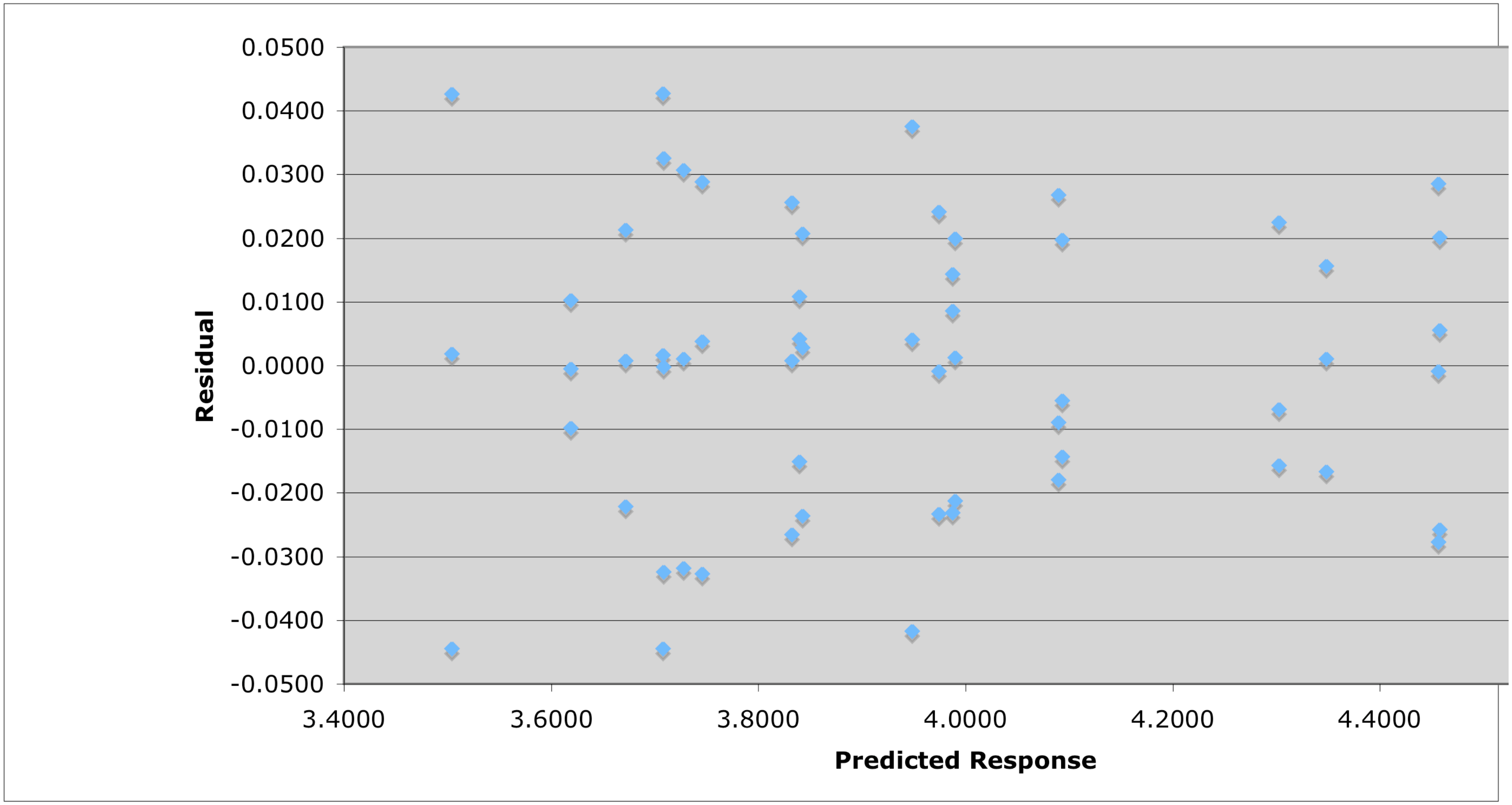
- **90% confidence interval for interaction effect**
  - $\gamma_{ij} \pm z_{.95}\, s_{\gamma_{ij}}$

$$s_{\gamma_{ij}} = s_e \sqrt{(a-1)(b-1)/abr}$$

Workload	W	X	Y	Z
I	(-0.0411 , -0.0013)	(-0.0043 , 0.0354)	(-0.0166 , 0.0231)	(-0.0174 , 0.0223)
J	(0.0200 , 0.0598)	(0.0134 , 0.0532)	(-0.1267 , -0.0870)	(0.0138 , 0.0535)
K	(-0.0645 , -0.0248)	(0.0276 , 0.0673)	(-0.0249 , 0.0148)	(-0.0176 , 0.0222)
L	(0.0366 , 0.0763)	(-0.1366 , -0.0969)	(0.0855 , 0.1252)	(-0.0649 , -0.0252)
M	(-0.0503 , -0.0106)	(0.0006 , 0.0404)	(-0.0165 , 0.0232)	(-0.0132 , 0.0265)

  - pink cells are not significant

Prediction Error for Code Size Study



Quantile-Quantile Plot for Code Size Study