An Extension of Chaiken's Algorithm to B-Spline Curves with Knots in Geometric Progression

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Abstract

Chaiken's algorithm is a procedure for inserting new knots into uniform quadratic B-spline curves by doubling the control points and taking two successive averages. Lane and Riesenfeld showed that Chaiken's algorithm extends to uniform B-spline curves of arbitrary degree. By generalizing the notion of successive averaging, we further extend Chaiken's algorithm to B-spline curves of arbitrary degree for knot sequences in geometric and affine progression.

1 Subdivision for knots in arithmetic progression

Let $N_k^{n+1}(t)$ be the B-spline basis function of degree $n+1$ whose support lies over the knot sequence $t_{2k}, t_{2k+2}, ..., t_{2k+2n+4}$ and let $\hat{N}_k^{n+1}(t)$ be the B-spline basis function of degree $n+1$ whose support lies over the refined knot sequence $t_k, t_{k+1}, ..., t_{k+n+2}$. Since the B-splines form a basis, there exist constants α_j^{n+1} such that

$$N_k^{n+1}(t) = \sum_{j=0}^{n+2} \alpha_j^{n+1} \hat{N}_{j+2k}^{n+1}(t). \tag{1}$$

For the degree zero basis functions, the $\alpha's$ satisfy

$$\alpha_0^0 = \alpha_1^0 = 1, \tag{2}$$

but for arbitrary knot sequences, the α's depend on k.

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In the case of uniform (arithmetic) knot sequences, Lane and Riesenfeld [LR80, Rie75] observed that the α_j^{n+1}'s are independent of k. For knot sequences satisfying $t_{i+1} = t_i + \gamma$, the B-spline basis functions satisfy the identities

$$\begin{aligned} N_0^{n+1}(t - 2k\gamma) &= N_k^{n+1}(t), \\ \hat{N}_0^{n+1}(t - k\gamma) &= \hat{N}_k^{n+1}(t). \end{aligned}$$

If equation 1 holds for $k = 0$, then for any k

$$\begin{aligned} N_k^{n+1}(t) &= N_0^{n+1}(t - 2k\gamma), \\ &= \sum_{j=0}^{n+2} \alpha_j^{n+1} \hat{N}_j^{n+1}(t - 2k\gamma) \\ &= \sum_{j=0}^{n+2} \alpha_j^{n+1} \hat{N}_{j+2k}^{n+1}(t). \end{aligned}$$

Therefore in the uniform case, any formula for subdividing N_0^n is automatically a formula for subdividing N_k^n. Lane and Riesenfeld [LR80] then observed that the following recurrence holds among the α's.

Theorem 1 *For any knot sequence satisfying $t_{i+1} = t_i + \gamma$, the α's satisfy the recurrence*

$$\alpha_j^{n+1} = \frac{1}{2}\alpha_{j-1}^n + \frac{1}{2}\alpha_j^n. \tag{3}$$

This recurrence leads directly to a subdivision (knot insertion) algorithm for B-spline curves with knots in arithmetic progression. To illustrate this algorithm, consider the case of a quadratic B-spline curve with a single nonzero control point P_k

$$S(t) = P_k N_k^2(t).$$

Recurrence 3 can be used to compute the new non-zero control points Q_k, Q_{k+1}, Q_{k+2}, and Q_{k+3} of the subdivided B-spline curve satisfying

$$S(t) = Q_k \hat{N}_k^2(t) + Q_{k+1}\hat{N}_{k+1}^2(t) + Q_{k+2}\hat{N}_{k+2}^2(t) + Q_{k+3}\hat{N}_{k+3}^2(t).$$

Figure 1 illustrates the diagrams for three separate recurrences starting at P_0, P_1, and P_2. In the degree zero case, the α^0's satisfy $\alpha_0^0 = \alpha_1^0 = 1$. Therefore, the topmost level of the diagrams contain two copies of P_k. By equation 3, the α^{n+1}'s can be computed from α^n's via averaging. Therefore, control points at $(n+1)$st level of the diagrams can be computed

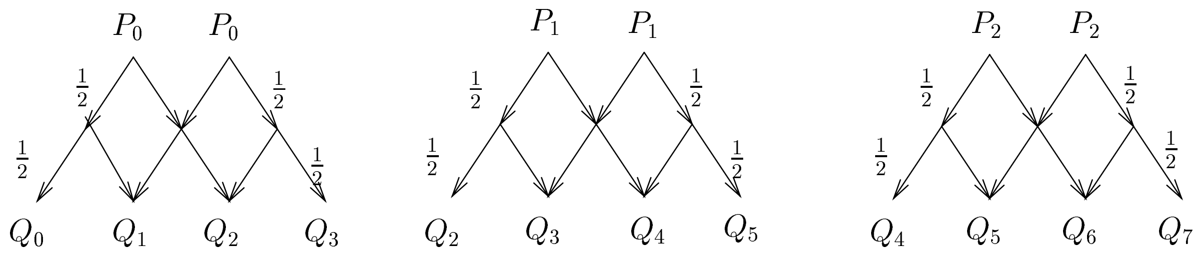

FIGURE 1: Subdivision recurrences for single basis functions

from control points on the nth level of the diagrams via averaging. Edges in the diagrams are labeled by an associated multiplier $\frac{1}{2}$. The control points for the subdivided B-spline curve are produced on the bottom level of the diagram.

Given a B-spline curve with control points P_k

$$S(t) = \sum_k P_k N_k^{n+1}(t),$$

the Lane-Riesenfeld algorithm computes new control points Q_k over a refined knot sequence that defines the same curve

$$S(t) = \sum_k Q_k \hat{N}_k^{n+1}(t).$$

In general, several adjacent control points in the original B-spline curve may contribute to the same control point in the subdivided B-spline curve. To combine these contributions, the recurrence for each individual basis function can be overlapped to form a single global recurrence for all the basis functions (see figure 2). In the quadratic case, the resulting method is exactly Chaiken's algorithm [Cha74, Rie75].

This paper investigates other types of knot sequences for which the corresponding B-spline curves have analogous subdivision algorithms. In particular, we show that B-spline curves defined over knot sequences in geometric progression possess a similar subdivision

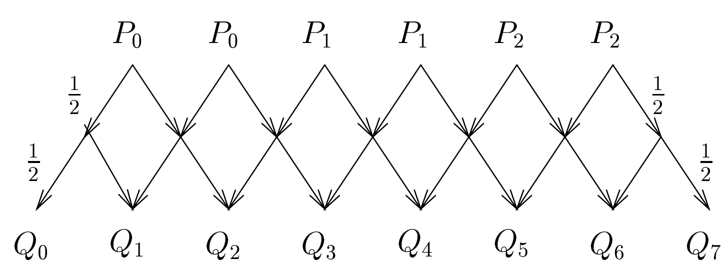

FIGURE 2: Chaiken's algorithm

algorithm. In the process of proving this algorithm, a new proof of the Lane-Riesenfeld algorithm is derived.

2 Subdivision for knots in geometric progression

A knot sequence is in geometric progression if $t_{i+1} = \beta t_i$ where $\beta \neq 1$. For geometric knot sequences, the basis functions are related by the identities

$$\begin{aligned} N_0^{n+1}(t/\beta^{2k}) &= N_k^{n+1}(t), \\ \hat{N}_0^{n+1}(t/\beta^{k}) &= \hat{N}_k^{n+1}(t). \end{aligned}$$

If equation 1 holds for $k = 0$, then for any k

$$\begin{aligned} N_k^{n+1}(t) &= N_0^{n+1}(t/\beta^{2k}), \\ &= \sum_{j=0}^{n+2} \alpha_j^{n+1} \hat{N}_j^{n+1}(t/\beta^{2k}) \\ &= \sum_{j=0}^{n+2} \alpha_j^{n+1} \hat{N}_{j+2k}^{n+1}(t). \end{aligned}$$

Therefore, any subdivision formula for N_0^{n+1} is automatically a subdivision formula for N_k^{n+1}. In the case of knots in geometric progression, the α's of equation 1 can be computed using the following theorem.

Theorem 2 *For knot sequences satisfying $t_{i+1} = \beta t_i$, the α's satisfy the recurrence*

$$\alpha_j^{n+1} = \frac{\beta^{n+1}}{1+\beta^{n+1}}\alpha_{j-1}^{n} + \frac{1}{1+\beta^{n+1}}\alpha_j^{n}. \tag{4}$$

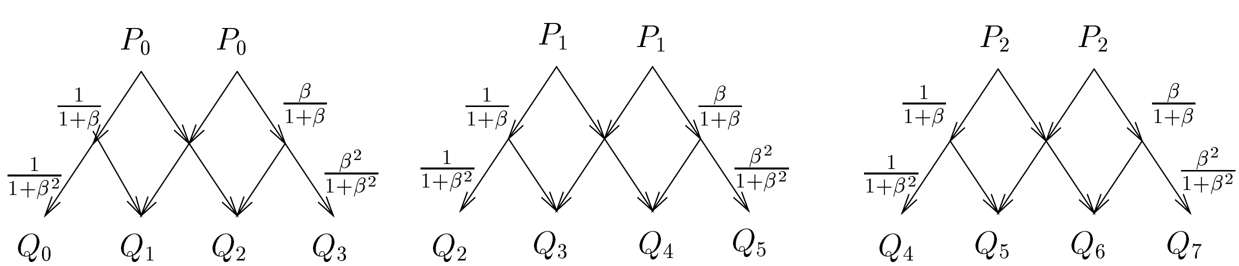

FIGURE 3: Subdivision recurrences for single basis functions

A subdivision algorithm similar to the Lane-Riesenfeld algorithm can now be derived in manner similar to that of the previous section. Applying Theorem 2 to a single basis function $P_k N_k^n(t)$ yields the recurrences shown in figure 3. Note that the multipliers on the kth level of the diagram are $\frac{\beta^k}{1+\beta^k}$ and $\frac{1}{1+\beta^k}$. Contributions of neighboring basis functions from the original B-spline curve to the same basis function in the subdivided B-spline curve can be computed by overlapping the recurrences as shown in figure 4.

More generally, if the B-spline curve is of the form

$$S(t) = \sum_k P_k N_k^{n+1}(t),$$

then the subdivision algorithm proceeds as follows:

1. Construct a new sequence of control points $Q_{2k}^0 = Q_{2k+1}^0 = P_k$.

2. For $j = 1$ to $n+1$,

   (a) For all control points Q_k^j, let

$$Q_k^j = \frac{\beta^j}{1+\beta^j} Q_k^{j-1} + \frac{1}{1+\beta^j} Q_{k+1}^{j-1}.$$

The resulting control points now satisfy

$$S(t) = \sum_k Q_k^{n+1} \hat{N}_k^{n+1}(t).$$

3 A proof using Prautsch's method

To prove Theorems 1 and 2, we use a variant of Prautsch's proof of the Oslo algorithm [Pra84]. The key to this proof is to apply the subdivision recurrence of equation 1 and the Cox-de Boor degree recurrence to $N_k^{n+1}(t)$. Two different expressions result, depending on

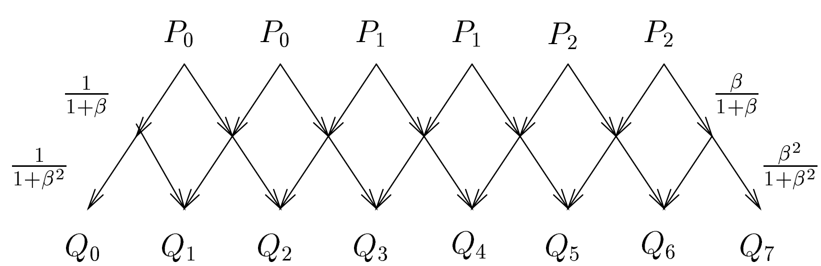

FIGURE 4: A subdivision method for knots in geometric progression

the order in which the two equations are applied to $N_k^{n+1}(t)$. The necessary degree recurrence is found in [dB72].

$$N_k^{n+1}(t) = \frac{t - t_{2k}}{t_{2k+2n+2} - t_{2k}} N_k^n(t) + \frac{t_{2k+2n+4} - t}{t_{2k+2n+4} - t_{2k+2}} N_{k+1}^n(t). \tag{5}$$

The proof will proceed over an arbitrary knot sequence with specialization to particular knot sequences deferred till appropriate.

First, we apply the subdivision formula of equation 1 to N_k^n and N_{k+1}^n in equation 5.

$$N_k^{n+1}(t) = \sum_{j=0}^{n+2} \left(\frac{t - t_{2k}}{t_{2k+2n+2} - t_{2k}} \alpha_j^n + \frac{t_{2k+2n+4} - t}{t_{2k+2n+4} - t_{2k+2}} \alpha_{j-2}^n \right) \hat{N}_{j+2k}^n(t). \tag{6}$$

Next, we apply the degree recurrence of equation 5 to the basis function $\hat{N}_j^n$ of equation 1.

$$N_k^{n+1}(t) = \sum_{j=0}^{n+2} \left(\frac{t - t_{2k+j}}{t_{2k+j+n+1} - t_{2k+j}} \alpha_j^{n+1} + \frac{t_{2k+j+n+1} - t}{t_{2k+j+n+1} - t_{2k+j}} \alpha_{j-1}^{n+1} \right) \hat{N}_{j+2k}^n(t). \tag{7}$$

Comparing the coefficients of $\hat{N}_j^n(t)$ in equations 6 and 7 yields the following relation among the α's.

$$\frac{t - t_{2k}}{t_{2k+2n+2} - t_{2k}} \alpha_j^n + \frac{t_{2k+2n+4} - t}{t_{2k+2n+4} - t_{2k+2}} \alpha_{j-2}^n = \frac{t - t_{2k+j}}{t_{2k+j+n+1} - t_{2k+j}} \alpha_j^{n+1} + \frac{t_{2k+j+n+1} - t}{t_{2k+j+n+1} - t_{2k+j}} \alpha_{j-1}^{n+1}. \tag{8}$$

This is the fundamental recurrence from which proofs of both theorems will be derived.

3.1 Knots in arithmetic progression

The Lane-Riesenfeld algorithm works for knots in arithmetic progression, $t_{i+1} = t_i + \gamma$. If we assume for the sake of simplicity that $t_0 = 0$ and $\gamma = 1$, then sequence satisfies $t_i = i$. Equation 8 simplifies to

$$\frac{t - 2k}{2n + 2} \alpha_j^n + \frac{2k + 2n + 4 - t}{2n + 2} \alpha_{j-2}^n = \frac{t - (2k + j)}{n + 1} \alpha_j^{n+1} + \frac{2k + j + n + 1 - t}{n + 1} \alpha_{j-2}^{n+1}.$$

Since this equality holds for all t, the t coefficients must be equal.

$$\frac{\alpha_j^n}{2} - \frac{\alpha_{j-2}^n}{2} = \alpha_j^{n+1} - \alpha_{j-1}^{n+1}. \tag{9}$$

Note that this recurrence is independent of k.

Equation 3 can be derived from equation 9 by induction on j. For $j = 0$, equation 9 simplifies to

$$\alpha_0^{n+1} = \frac{\alpha_0^n}{2}.$$

If equation 3 holds for $j-1$, then

$$\alpha_{j-1}^{n+1} = \frac{\alpha_{j-2}^n}{2} + \frac{\alpha_{j-1}^n}{2}.$$

Substituting this expression into equation 9 yields

$$\alpha_j^{n+1} = \frac{\alpha_j^n}{2} - \frac{\alpha_{j-2}^n}{2} + (\frac{\alpha_{j-2}^n}{2} + \frac{\alpha_{j-1}^n}{2}) = \frac{\alpha_j^n}{2} + \frac{\alpha_{j-1}^n}{2}.$$

Therefore, equation 3 holds for all j.

3.2 Knots in geometric progression

Equation 8 is also crucial in proving Theorem 2 correct. If the knots are in geometric progression, $t_{i+1} = \beta t_i$; setting $t_0 = 1$ yields that $t_i = \beta^i$. After simplification, the constant terms of equation 8 satisfy the relation:

$$-\frac{1}{1+\beta^{n+1}}\alpha_j^n + \frac{\beta^{2n+2}}{1+\beta^{n+1}}\alpha_{j-2}^n = -\alpha_j^{n+1} + \beta^{n+1}\alpha_{j-1}^{n+1}. \tag{10}$$

Again, equation 4 can be derived from equation 10 by induction on j. For $j = 0$, the formulas trivially agree. If equation 4 holds for $j-1$, then

$$\alpha_{j-1}^{n+1} = \frac{\beta^{n+1}}{1+\beta^{n+1}}\alpha_{j-2}^n + \frac{1}{1+\beta^{n+1}}\alpha_{j-1}^n.$$

Substituting this expression into equation 10 yields

$$\begin{aligned}\alpha_j^{n+1} &= \beta^{n+1}(\frac{\beta^{n+1}}{1+\beta^{n+1}}\alpha_{j-2}^n + \frac{1}{1+\beta^{n+1}}\alpha_{j-1}^n) + \frac{1}{1+\beta^{n+1}}\alpha_j^n - \frac{\beta^{2n+2}}{1+\beta^{n+1}}\alpha_{j-2}^n, \\ &= \frac{\beta^{n+1}}{1+\beta^{n+1}}\alpha_{j-1}^n + \frac{1}{1+\beta^{n+1}}\alpha_j^n.\end{aligned}$$

Therefore, equation 4 holds for all j.

Applying a similar approach to the t terms of equation 8 yields a second recurrence among the α's.

$$\alpha_j^{n+1} = \frac{\beta^j}{1+\beta^{n+1}}\alpha_j^n + \frac{\beta^{j-1}}{1+\beta^{n+1}}\alpha_{j-1}^n.$$

Combining this equation and equation 4 yields a recurrence relating α_j^n to α_{j-1}^n:

$$\alpha_j^n = \frac{\beta^{j-1}(1-\beta^{n-j+2})}{1-\beta^j}\alpha_{j-1}^n.$$

This formula in conjunction with the fact that

$$\alpha_0^n = \prod_{k=1}^{n}\frac{1}{1+\beta^k}$$

allows for fast explicit construction of the α's.

4 Subdivision for knots in affine progression

Arithmetic and geometric progressions may be viewed as special instances of a more general type of sequence, the affine progression:

$$t_{i+1} = \alpha t_i + \delta$$

A simple example of an affine progression is the sequence 0, $\frac{1}{2}$, $\frac{3}{4}$, $\frac{7}{8}$, $\frac{15}{16}$, ... This sequence satisfies the relation $t_{i+1} = \frac{1}{2}t_i + \frac{1}{2}$.

If the original knot sequence t_0, t_2, t_4, t_6, ... satifies the relation

$$t_{i+2} = \beta^2 t_i + 2\gamma,$$

then for $\beta = 1$ the resulting t_i lie in arithmetic progression. For $\gamma = 0$, the t_i lie in geometric progression. This knot sequence can be refined to create a new knot sequence t_0, t_1, t_2, t_3, ... in affine progression via the following relation:

$$t_{i+1} = \beta t_i + \frac{2\gamma}{1+\beta} \tag{11}$$

Note that t_0, t_2, t_4, ... still lie in the original affine progression.

$$\begin{aligned} t_{i+2} &= \beta t_{i+1} + \frac{2\gamma}{1+\beta}, \\ &= \beta(\beta t_i + \frac{2\gamma}{1+\beta}) + \frac{2\gamma}{1+\beta}, \\ &= \beta^2 t_i + 2\gamma(\frac{\beta}{1+\beta} + \frac{1}{1+\beta}), \\ &= \beta^2 t_i + 2\gamma. \end{aligned}$$

When $\beta \neq 1$, affine progressions and geometric progressions are closely related. Specifically, the elements of an affine progression can be translated to bring the sequence into a geometric progression. Given the affine progression of equation 11, we define a translate of this sequence as follows:

$$\hat{t}_i = t_i + \frac{2\gamma}{\beta^2 - 1}. \tag{12}$$

This new sequence of $\hat{t}_i$'s lies in geometric progression since

$$\begin{aligned} \hat{t}_{i+1} &= t_{i+1} + \frac{2\gamma}{\beta^2 - 1}, \\ &= \beta t_i + \frac{2\gamma}{1+\beta} + \frac{2\gamma}{\beta^2 - 1}, \end{aligned}$$

$$
\begin{aligned}
&= \beta(t_i + \frac{2\gamma}{\beta^2 - 1}), \\
&= \beta\hat{t}_i.
\end{aligned}
$$

Because of this observation, the subdivision algorithm of section 2 can be applied to B-spline curves whose knots are in affine progression. For $\beta = 1$, the resulting knots are in arithmetic progression. Substituting $\beta = 1$ into the recurrence 4 yields the Lane-Riesenfeld recurrence of equation 3. Thus, the subdivision algorithm of section 2 degenerates into the Lane-Riesenfeld algorithm for knots in arithmetic progression. For $\beta \neq 1$, the B-spline curve may be reparameterized via equation 12 to bring the knots into geometric progression. Since this reparameterization is translation, it does not affect the geometry of the B-spline curve. After applying the subdivision algorithm of section 2, equation 11 may be used to compute the new refined knot sequence.

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