COMP 182 Algorithmic Thinking

Counting

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Reading Material

- Chapter 6, Sections 1-6
- Chapter 8, Section 5

- Counting is a branch of combinatorics that is concerned with enumerating objects with certain properties.
- We have already seen an important area where counting is essential: Complexity of algorithms.
- Another area where counting is essential is probability (our next topic).
- Counting is also important in designing codes, in understanding complexity of systems,…

The Product Rule

- If a procedure can be broken down into a sequence of two tasks such that there are n_1 ways to do the first task and for each of these ways there are n_2 ways to do the second task, then there are $n_1 n_2$ ways to do the procedure.
- This can be generalized to procedures that can be broken down into a sequence of k tasks.

The Product Rule

- How many different DNA sequences of length 10 are there?
- How many 1-1 functions are there from a set with m elements to a set of n elements?

The Product Rule

- The product rule is analogous to the size of the cartesian product of sets:

$$|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \cdot |A_2| \cdot \cdots \cdot |A_m|.$$

The Sum Rule

- If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are n_1+n_2 ways to do the task.
- This can be generalized to more than two categories.

The Sum Rule

$$
\begin{aligned}
&k := 0\\
&\textbf{for } i_1 := 1 \textbf{ to } n_1\\
&\qquad k := k + 1\\
&\textbf{for } i_2 := 1 \textbf{ to } n_2\\
&\qquad k := k + 1\\
&\qquad\qquad \vdots\\
&\textbf{for } i_m := 1 \textbf{ to } n_m\\
&\qquad k := k + 1
\end{aligned}
$$

What's the value of k after this piece of code is executed?

The Sum Rule

- The sum rule is analogous to the size of the union of pairwise disjoint sets:

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m| \text{ when } A_i \cap A_j = \emptyset \text{ for all } i,\ j.$$

The Sum Rule

- A valid password is a sequence between six and eight symbols.
- The first symbol must be a letter (lowercase or uppercase) and the remaining symbols must be either letters or digits.
- What is the number of different possible passwords?

The Subtraction Rule

- If a task can be done either in one of n_1 ways or in one of n_2 ways, where m ways are shared between the two categories, then there are n_1+n_2-m ways to do the task.

The Subtraction Rule

- The subtraction rule is analogous to the size of the union of two sets:

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$

The Subtraction Rule

- How many n-element DNA sequences start with T or end with G?

Product, Summation, and Subtraction

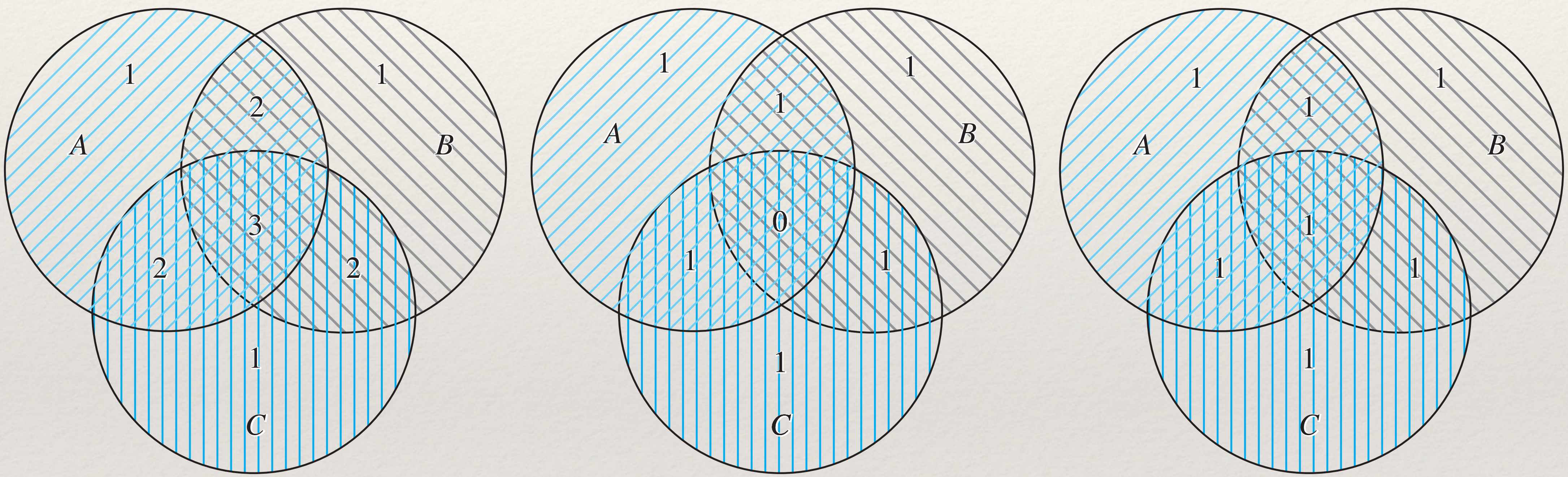
- Suppose that a password could be at least 4 characters long and at most 8 characters long, and each character is either a letter (lower or upper case) or a digit. Furthermore, a password must contain at least one of the special symbols %,$,#,@. How many passwords are there?

The Principle of Inclusion-Exclusion

- Generalizes the subtraction rule to any finite number of finite sets.

The Principle of Inclusion-Exclusion

- Illustrating the size of the union of three sets



(a) Count of elements by $|A|+|B|+|C|$

(b) Count of elements by $|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|$

(c) Count of elements by $|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C|$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

The Principle of Inclusion-Exclusion

THE PRINCIPLE OF INCLUSION–EXCLUSION Let $A_1, A_2, \ldots, A_n$ be finite sets. Then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|.$$

The Division Rule

- There are n/d ways to do a task if it can be done using a procedure that can be carried out in n ways, and for every way w, exactly d of the n ways correspond to way w.

The Division Rule

- How many ways are there to seat n people around a circular table where two seatings are considered the same when everyone has the same immediate left and immediate right neighbor?

Tree Diagrams

- To use trees in counting, we use a branch to represent each possible choice.
- We represent the possible outcomes by the leaves.

Tree Diagrams

- How many bit strings of length four do not have two consecutive 1s?

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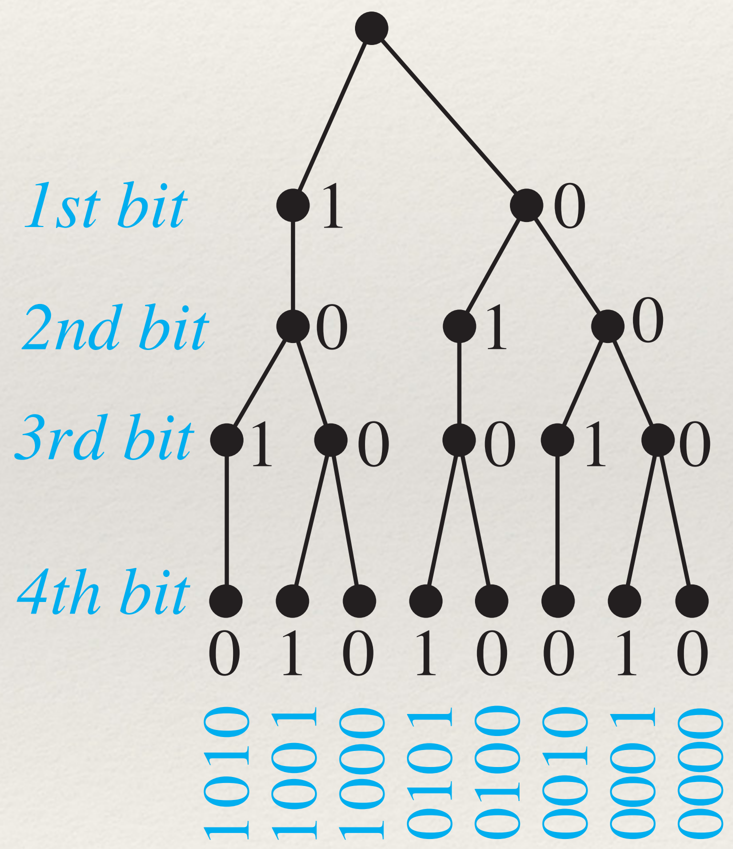

Tree Diagrams

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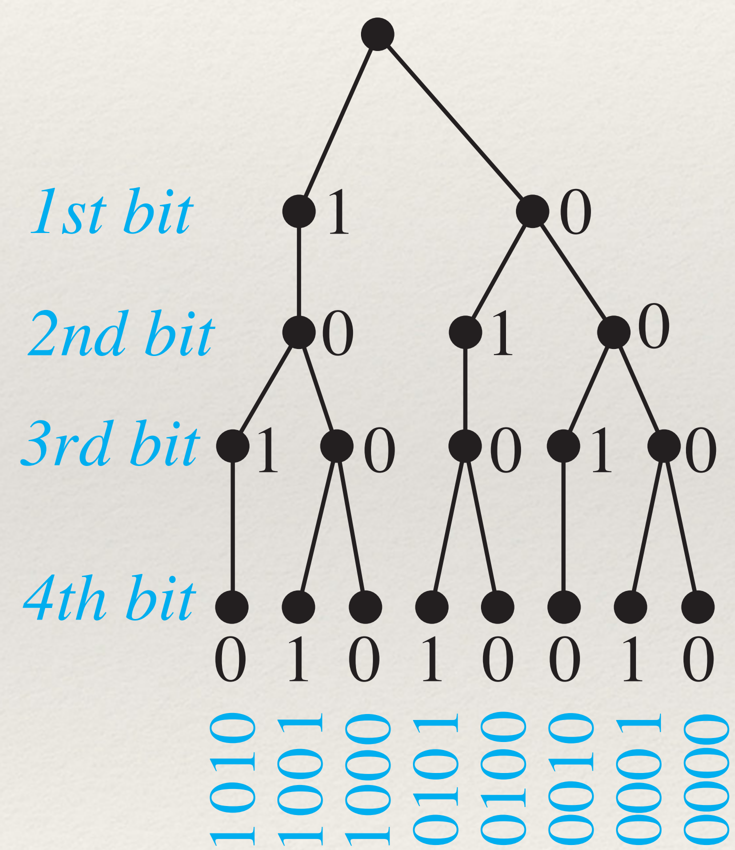

This can be viewed as a finite automaton whose start state is the root and whose accepting states are the leaves!

The Pigeonhole Principle

- If k is a positive integer and $k+1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

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Example: 13 pigeons and 12 pigeonholes

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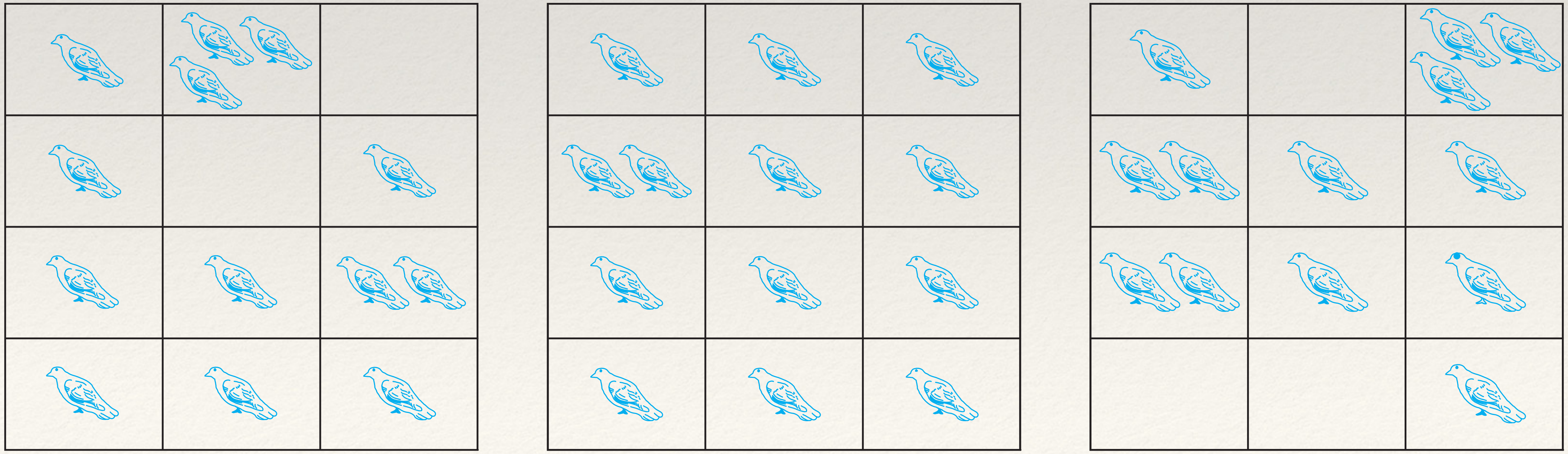


The Pigeonhole Principle

- More generally, if N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

The Pigeonhole Principle

- Prove: In any graph there are (at least) two nodes with the same degree.

Permutations and Combinations

- Let n be a positive integer and r be an integer with $0 \leq r \leq n$.
- Denote by $P(n,r)$ the number of all r-permutations of a set with n distinct elements.
- Denote by $C(n,r)$ the number of all r-combinations of a set with n distinct elements.

$$P(n,r) = \frac{n!}{(n-r)!} \qquad C(n,r) = \frac{n!}{r!(n-r)!}$$

Permutations and Combinations

- Notice that $P(n,r)$ and $C(n,r)$ are related by the division rule.

Permutations and Combinations

- How many bit strings of length 12 contain
  - exactly three 1s?
  - at most three 1s?
  - at least three 1s?
  - an equal number of 0s and 1s?

Binomial Coefficients

THE BINOMIAL THEOREM Let x and y be variables, and let n be a nonnegative integer. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

Binomial Coefficients

- Results that are straightforward to establish using the binomial theorem:

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k}$$

$$0 = ((-1)+1)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k}$$

$$3^n = (1+2)^n = \sum_{k=0}^{n} 2^k \binom{n}{k}$$

Binomial Coefficients

- Important identities:

Pascal's identity
$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Vandermonde's identity
$$\binom{m+n}{k} = \sum_{\ell=0}^{k} \binom{m}{k-\ell}\binom{n}{\ell}$$

$$\binom{n}{k} = \binom{n}{n-k}$$

Generalized Permutations and Combinations

- How many bit strings of length r are there?

Generalized Permutations and Combinations

- The number of r-permutations of a set of n objects with repetition allowed is n^r.

Generalized Permutations and Combinations

- How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 17$$

where x_1, x_2, x_3, and x_4 are nonnegative integers?

Generalized Permutations and Combinations

- The number of r-combinations of a set of n objects with repetition allowed is $C(n+r-1,r)$.

Generalized Permutations and Combinations

$k := 0$
for $i_1 := 1$ **to** n
    for $i_2 := 1$ **to** i_1
        .
        .
        .
            for $i_m := 1$ **to** i_{m-1}
                $k := k + 1$

What's the value of k after this piece of code is executed?

Generalized Permutations and Combinations

- The multinomial theorem: Let n be a nonnegative integer and m be a positive integer. Then,

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{r_1 + r_2 + \cdots + r_m = n} \binom{n}{r_1, r_2, \ldots, r_m} \prod_{1 \le \ell \le m} x_\ell^{r_\ell}$$

where

$$\binom{n}{r_1, r_2, \ldots, r_m} = \frac{n!}{r_1! r_2! \cdots r_m!}$$

The multinomial coefficient: the number of ways to distribute n objects into m boxes with r_i objects in box i.

Combinatorial Proofs

- A <u>combinatorial proof</u> of an identity is a proof that
  - uses <u>counting arguments</u> to prove that both sides of the identity count the same objects but in different ways, *or*
  - is based on showing that there is a <u>bijection</u> between the sets of objects counted by the two sides of the identity.

Combinatorial Proofs

Prove:

$$\sum_{r=0}^{n} \binom{n}{r} \binom{2n}{n-r} = \binom{3n}{n}$$

A Word (or, few words) on the Cardinality of Sets

- Section 2.5 in your textbook

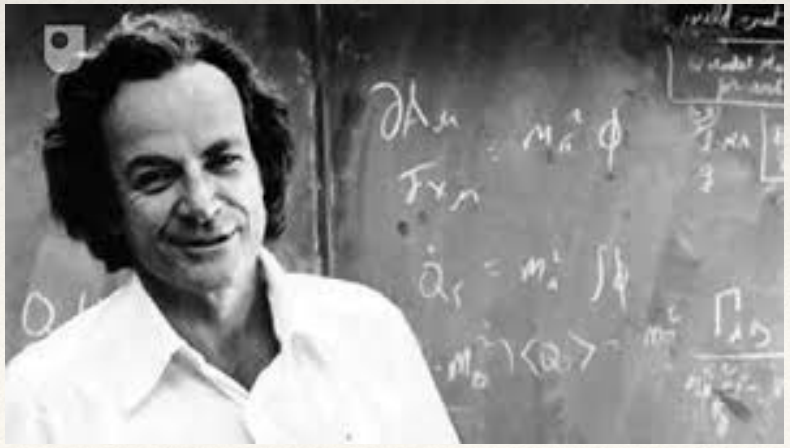
Finite Sets

- We can list all the elements of a finite set.
- We can order them so that we can speak of the first element, the second element,…, and the last element.

Infinity

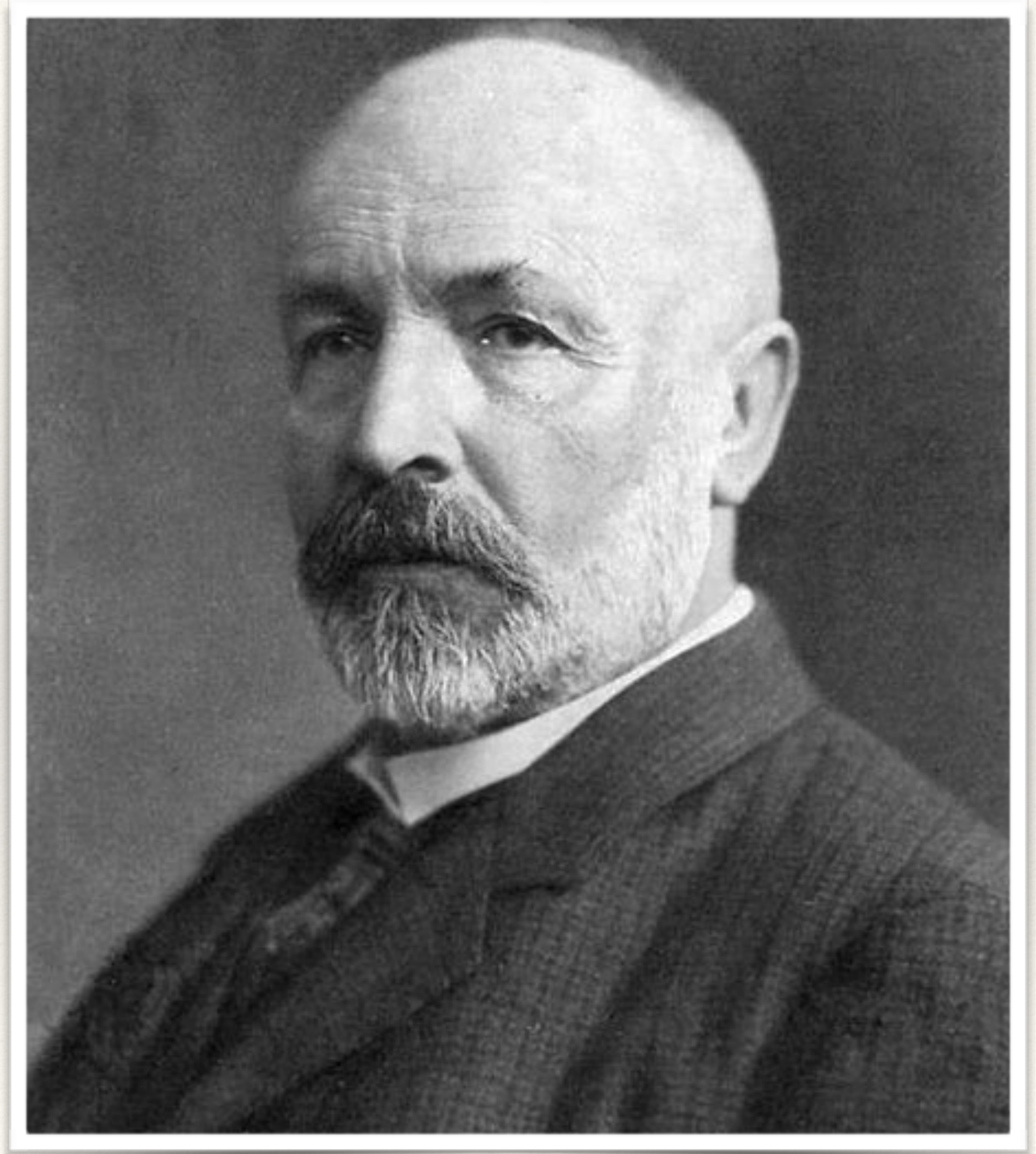
- Think of listing the items of a set.
- You can speak of the first item, the second item,…
- You cannot speak of the last item.

- “It is my task to convince you not to turn away because you don’t understand it. You see, my physics students don’t understand it either. That is because I don’t understand it. Nobody does.”



Georg Cantor

- Studied infinity and classified it.
- He suffered a series of breakdowns.
- Just a coincidence or is there causation?!?!



To Infinity and Beyond

- Philosophers, religious people and many mathematicians had linked infinity to God: The ultimate limit; there is nothing more, nothing greater.
- Cantor wanted to go beyond infinity.

The Birth of $\aleph_0$

- Cantor wanted to distinguish "real" infinity from the fuzzy notion of infinity denoted by ∞.
- He introduced $\aleph_0$ to denote the infinity that contains the set of natural numbers.

Non-numbers

- Please do not forget this for the rest of your life:
  - ∞, $\aleph_0$, $\cdots$ are **not** numbers!

Infinity: A Matter of Definition

- Cantor **defined** an infinite set as one that had a one-to-one correspondence with a subset of itself.

Infinity: A Matter of Definition

- In other words, cardinality is **not** about the number of items in a set; it **is** about having 1-1 correspondence with another set.

Cardinality of Sets

- Let A and B be two sets (not necessarily finite).
- If there exists a bijection (1-1 and onto function) from A to B, then A and B have the same cardinality ($|A| = |B|$).
- If there exists a 1-1 function from A to B, then $|A| \leq |B|$.

Countable Sets

- A set is <u>countable</u> if it is either finite or has the same cardinality as the set of positive integers.
- Examples of countable sets
  - the set of odd positive integers ($f(n)=2n-1$)
  - the set of all positive rational numbers
  - the set of all finite-length strings over a finite alphabet

Back to $\aleph_0$

- It is the cardinality of a countable infinite set.

An Algorithmic Perspective

- A set is countable if you can come up with an algorithm to enumerate it (enumerator).
- You have an enumerator if every element of the set is "printed" after a finite number of steps.

Enumerating the Positive Rationals

$$
\begin{array}{cccccc}
\frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \frac{5}{1} & \cdots \\[1em]
\frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \frac{5}{2} & \cdots \\[1em]
\frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \frac{5}{3} & \cdots \\[1em]
\frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \frac{5}{4} & \cdots \\[1em]
\frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & \frac{5}{5} & \cdots \\[1em]
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

Enumerating All Integers

- n=1;
- While True
    - If n is even
        - print n/2;
    - Else
        - print -(n-1)/2;
    - n=n+1;

Uncountable Sets

- The set of real numbers is uncountable.
- The power-set of a countable infinite set is uncountable.
- A powerful tool for proving uncountability is <u>Cantor's diagonalization</u> technique.

Uncountable Sets

- The cardinality of $(-\pi/2, \pi/2)$ and the reals is the same.
- One bijection from the former to the latter is $tan(x)$.

Schroder-Bernstein Theorem

- If A and B are two sets with $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

Uncountable Sets

- The two sets (0,1) and (0,1] have the same cardinality.

$$
\begin{tikzcd}
(0,1) \arrow[r, shift left, "f(x)=x"] & (0,1] \arrow[l, shift left, "g(x)=x/2"]
\end{tikzcd}
$$

The Continuum Hypothesis

- An important theorem of Cantor's states that the cardinality of a set is always less than the cardinality of its power set.
- Corollary: The set of positive integers has a smaller cardinality than the power set of the positive integers.
- We can show that the power set of the positive integers and the reals have the same cardinality.

The Continuum Hypothesis

- The continuum hypothesis asserts that there is no cardinal number between the cardinality of the positive integers and the cardinality of the reals.
- It can be shown that the smallest infinite cardinal numbers form an infinite sequence $\aleph_0 < \aleph_1 < \aleph_2 < \cdots$
- If we assume the hypothesis is true, then the cardinality of the reals is $\aleph_1$.

The Continuum Hypothesis

- The continuum hypothesis is still an open question.
- It has been shown that it can be neither proved nor disproved under the standard set theory axioms in modern mathematics, the Zermelo-Fraenkel axioms.
- Should this set of axioms be replaced by some other set of axioms for set theory? If you're intrigued, study math!

Questions?