Sets, Propositional Logic, Predicates, and Quantifiers
Reading Material

- Chapter 1, Sections 1, 4, 5
- Chapter 2, Sections 1, 2
Mathematics is about statements that are either true or false.

Such statements are called propositions.

We use logic to describe them, and proof techniques to prove whether they are true or false.
Propositions

❖ 5>7
❖ The square root of 2 is irrational.
❖ A graph is bipartite if and only if it doesn’t have a cycle of odd length.
❖ For $n>1$, the sum of the numbers 1,2,3,…,$n$ is $n^2$. 
Propositions?

- E=mc²
- The sun rises from the East every day.
- All species on Earth evolved from a common ancestor.
- God does not exist.
- Everyone eventually dies.
And some of you might already be wondering: “If I wanted to study mathematics, I would have majored in Math. I came here to study computer science.”
Computer Science *is* mathematics, but we almost exclusively focus on aspects of mathematics that relate to computation (that can be implemented in software and/or hardware).
Logic is the language of computer science and, mathematics is the computer scientist’s most essential toolbox.
Examples of “CS-relevant” Math

❖ Algorithm A correctly solves problem P.
❖ Algorithm A has a worst-case running time of $O(n^3)$.
❖ Problem P has no solution.
❖ Using comparison between two elements as the basic operation, we cannot sort a list of $n$ elements in less than $O(n \log n)$ time.
❖ Problem A is NP-Complete.
“Algorithm A is correct” is a proposition that requires a mathematical proof.

All students in the course thinking that it is true is not a proof.

Showing it is true on 1 million examples is not a proof.
“Problem P has no solution” is a proposition that requires a mathematical proof.

Your inability to come up with a solution to Problem P is not a proof that a solution doesn’t exist.

All your 5,000 Facebook friends not being able to come up with a solution doesn’t make the statement true either.
Despite decades of work by so many brilliant researchers, no one has been able to come up with a polynomial-time algorithm for the Traveling Salesman Problem (TSP).

Still, no computer scientist or mathematician would state “TSP has no polynomial-time solution” because such a statement would require a mathematical proof and such a proof has not been found yet.
It is important to note that decades of work by brilliant researchers not resulting in a polynomial-time algorithm for TSP do strengthen our belief that the conjecture that “TSP has no polynomial-time solution” is true.

This belief could, for example, direct other brilliant researchers to focus on proving the conjecture true (rather than false).

However, no matter how strong our belief is, it is still not a proof.
Sets

❖ A set is an unordered collection of items.

❖ We write \( a \in S \) to denote that \( a \) is an element of set \( S \), or that set \( S \) contains element \( a \).

❖ Roster method description of sets: \( B = \{0,1\} \), \( C = \{a,b,c,d\} \), \( D = \{\#,\$,\%,\&,\@\} \)

❖ Set builder or set comprehension description of sets: \( F = \{x \mid x \text{ is an odd integer}\} \), \( G = \{y \mid y \text{ is an integer that is divisible by } 7\} \)
Sets

- An element of a set cannot appear more than once in the set.
- For example, \{a,b,b,c\} is \textit{not} a set.
- A mathematical structure that allows for an element to appear more than once is called \textit{multiset} or \textit{bag}. In this course, we will only work with sets.
Special Sets

- The set of natural numbers \( \mathbb{N} = \{0, 1, 2, 3, \ldots \} \)
- The set of integers \( \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots \} \)
- The set of positive integers \( \mathbb{Z}^+ = \{1, 2, \ldots \} \)
- The set of rational numbers \( \mathbb{Q} = \{ \frac{p}{q} | p, q \in \mathbb{Z}, q \neq 0 \} \)
- The set of real numbers \( \mathbb{R} \)
- The set of positive real numbers \( \mathbb{R}^+ \)
The Empty Set

- The **empty set** is the set that contains no elements.
- Denoted by $\emptyset$ or {}.
- Important: The set $\{\emptyset\}$ is *not* empty. Rather, it is a set that contains one element that is $\emptyset$. 
Cardinality of Sets

❖ The **cardinality** of a finite set $S$, denoted by $|S|$, is the number of elements in $S$.

❖ $|\{a,b,c\}| = 3$

❖ $|\emptyset| = 0$

❖ $|\{\emptyset\}| = 1$
Cardinality of Sets

❖ Not all sets are finite.
❖ Infinite sets can be countable or uncountable.
❖ More on this later in the semester.
Propositional Logic
Propositions

- A proposition is a declarative sentence that is either true or false, but not both.

- We use propositional variables (e.g., \( p, q, r, s, \ldots \)) to represent propositions.
Propositions

❖ Propositions:
❖ \( 3 \in \{1,2,4\} \)
❖ \(|\{0,1\}| = 2\)
❖ \(7 \notin \{a,b,c\}\)

❖ Not propositions:
❖ \(1+1\)
❖ \(\{a,b,c\}\)
❖ \(|\{5,12,19\}|\)
Compound Propositions

- If $p$ is a proposition, $\neg p$ is its negation.
- If $p$ and $q$ are two propositions, then
  - $p \land q$ ("$p$ and $q$") is their conjunction
  - $p \lor q$ ("$p$ or $q$") is their disjunction
Truth Values

- The truth value of a proposition is true, denoted by $T$, if it is a true proposition, and the truth value is false, denoted by $F$, if it is a false proposition.

- True propositions:
  - $|\{a,b\}|=2$  $|\emptyset|<|\{1\}|$  $7\not\in\{1,5,9,12\}$

- False propositions:
  - $|\emptyset|=0$  $7\in\{1,5,9,12\}$
Truth Table

❖ For a compound proposition, one way to determine the truth value of the proposition is by using a truth table.
❖ The truth table has one row for each combination of T and F for the primitive propositions.
Truth Table

**TABLE 1** The Truth Table for the Negation of a Proposition.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$\neg p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

**TABLE 2** The Truth Table for the Conjunction of Two Propositions.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \land q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

**TABLE 3** The Truth Table for the Disjunction of Two Propositions.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \lor q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>
XOR, If, and Iff

**TABLE 4** The Truth Table for the Exclusive Or of Two Propositions.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \oplus q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

**TABLE 5** The Truth Table for the Conditional Statement $p \rightarrow q$.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \rightarrow q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

**TABLE 6** The Truth Table for the Biconditional $p \leftrightarrow q$.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \leftrightarrow q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>
EXAMPLE 10
Let \( p \) be the statement "You can take the flight," and let \( q \) be the statement "You buy a ticket." Then \( p \leftrightarrow q \) is the statement "You can take the flight if and only if you buy a ticket." This statement is true if \( p \) and \( q \) are either both true or both false, that is, if you buy a ticket and can take the flight or if you do not buy a ticket and you cannot take the flight. It is false when \( p \) and \( q \) have opposite truth values, that is, when you do not buy a ticket, but you can take the flight (such as when you get a free trip) and when you buy a ticket but you cannot take the flight (such as when the airline bumps you).

IMPLICIT USE OF BICONDITIONALS
You should be aware that biconditionals are not always explicit in natural language. In particular, the "if and only if" construction used in biconditionals is rarely used in common language. Instead, biconditionals are often expressed using an "if, then" or an "only if" construction. The other part of the "if and only if" is implicit. That is, the converse is implied, but not stated. For example, consider the statement in English "If you finish your meal, then you can have dessert." What is really meant is "You can have dessert if and only if you finish your meal." This last statement is logically equivalent to the two statements "If you finish your meal, then you can have dessert" and "You can have dessert only if you finish your meal." Because of this imprecision in natural language, we need to make an assumption whether a conditional statement in natural language implicitly includes its converse. Because precision is essential in mathematics and in logic, we will always distinguish between the conditional statement \( p \rightarrow q \) and the biconditional statement \( p \leftrightarrow q \).

Truth Tables of Compound Propositions
We have now introduced four important logical connectives—conjunctions, disjunctions, conditional statements, and biconditional statements—as well as negations. We can use these connectives to build up complicated compound propositions involving any number of propositional variables. We can use truth tables to determine the truth values of these compound propositions, as Example 11 illustrates. We use a separate column to find the truth value of each compound expression that occurs in the compound proposition as it is built up. The truth values of the compound proposition for each combination of truth values of the propositional variables in it is found in the final column of the table.

EXAMPLE 11
Construct the truth table of the compound proposition \((p \lor \neg q) \rightarrow (p \land q)\).

Solution:
Because this truth table involves two propositional variables \( p \) and \( q \), there are four rows in this truth table, one for each of the pairs of truth values TT, TF, FT, and FF. The first two columns are used for the truth values of \( p \) and \( q \), respectively. In the third column we find the truth value of \( \neg q \), needed to find the truth value of \( p \lor \neg q \), found in the fourth column. The fifth column gives the truth value of \( p \land q \). Finally, the truth value of \((p \lor \neg q) \rightarrow (p \land q)\) is found in the last column. The resulting truth table is shown in Table 7.

TABLE 7 The Truth Table of \((p \lor \neg q) \rightarrow (p \land q)\).

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( \neg q )</th>
<th>( p \lor \neg q )</th>
<th>( p \land q )</th>
<th>((p \lor \neg q) \rightarrow (p \land q))</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>
A tautology is a compound proposition whose truth value is T under all truth assignments to its propositional variables.

Examples:

- $p \lor \neg p$
- $(p \rightarrow q) \lor \neg q$
- $T \lor p$
Contradiction

A contradiction is a compound proposition whose truth value is F under all truth assignments to its propositional variables.

Examples:

- $p \land \neg p$
- $F \land \neg p$
Propositional Satisfiability

- A compound proposition is **satisfiable** if it is not a contradiction.
- A truth assignment to the propositional variables that make the compound proposition T is called a **solution** of this particular satisfiability problem.
Propositional Satisfiability

- The Propositional Satisfiability Problem, commonly known as SAT, is a decision problem that plays a central role in computer science.

- It is defined as:
  - Input: A compound proposition $\varphi$
  - Output: "Yes", if $\varphi$ is satisfiable, and "No" otherwise
Propositional Satisfiability

- A trivial algorithm for solving SAT would build the truth table of \( \varphi \) and check the rightmost column for a T.
- If \( \varphi \) has \( n \) propositional variables, how many rows does the truth table have?
- If the computer can build and evaluate 1000 rows a second, how many seconds does this algorithm take if \( n=10 \)? If \( n=1000 \)?
Propositional Satisfiability

Two seminal results:

❖ Stephen Cook (1971) showed that SAT is NP-Complete.

❖ Richard Karp (1972) introduced polynomial-time reductions as a tool to show other problems are NP-Complete.

❖ Both Cook and Karp won the Turing award for these contributions.
Predicate Logic
In mathematics and computer science, we often find statements that involve variables.

For example, \( x \in \{1,2,3\} \).

In this example, \( x \) is the variable, and "\( x \in \{1,2,3\} \)" is the predicate.
Predicate Logic

- The statement $x \in \{1, 2, 3\}$ can be denoted by propositional function $P(x)$.
- This propositional function evaluates to either T or F once a value has been assigned to variable $x$, in which case the statement $P(x)$ becomes a proposition.
- What is $P(1)$? $P(3)$? $P(7)$?
Predicate Logic

- These statements and functions may involve any number of variables.
- For example, $Q(x,y)$ is the statement “$x \in \{1,2,3\} \land y \notin \{a\}$”.
- What is the value of $Q(1,a)$? $Q(1,b)$?
Quantifiers

- Another way to turn a propositional function into a proposition is via quantification.
- Predicate calculus is the area of logic that deals with predicates and quantifiers.
Universal Quantification

The universal quantification of $P(x)$ is the statement “$P(x)$ for all values of $x$ in the domain of discourse” and is denoted by $\forall x P(x)$. 
Existential Quantification

❖ The existential quantification of $P(x)$ is the statement “$P(x)$ for some value of $x$ in the domain of discourse” and is denoted by $\exists x P(x)$. 
## Quantifiers

### TABLE 1

<table>
<thead>
<tr>
<th>Statement</th>
<th>When True?</th>
<th>When False?</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\forall x \ P(x)$</td>
<td>$P(x)$ is true for every $x$.</td>
<td>There is an $x$ for which $P(x)$ is false.</td>
</tr>
<tr>
<td>$\exists x \ P(x)$</td>
<td>There is an $x$ for which $P(x)$ is true.</td>
<td>$P(x)$ is false for every $x$.</td>
</tr>
</tbody>
</table>

**EXAMPLE 8**

Let $P(x)$ be the statement "$x + 1 > x$." What is the truth value of the quantification $\forall x P(x)$, where the domain consists of all real numbers?

**Solution:**

Because $P(x)$ is true for all real numbers $x$, the quantification $\forall x P(x)$ is true.

▲

**Remark:** Generally, an implicit assumption is made that all domains of discourse for quantifiers are nonempty. Note that if the domain is empty, then $\forall x P(x)$ is true for any propositional function $P(x)$ because there are no elements $x$ in the domain for which $P(x)$ is false.

**EXAMPLE 9**

Let $Q(x)$ be the statement "$x < 2$." What is the truth value of the quantification $\forall x Q(x)$, where the domain consists of all real numbers?

**Solution:**

$Q(x)$ is not true for every real number $x$, because, for instance, $Q(3)$ is false. That is, $x = 3$ is a counterexample for the statement $\forall x Q(x)$. Thus $\forall x Q(x)$ is false.

▲

**EXAMPLE 10**

Suppose that $P(x)$ is "$x^2 > 0$." To show that the statement $\forall x P(x)$ is false where the universe of discourse consists of all integers, we give a counterexample. We see that $x = 0$ is a counterexample because $x^2 = 0$ when $x = 0$, so that $x^2$ is not greater than 0 when $x = 0$.

▲

Looking for counterexamples to universally quantified statements is an important activity in the study of mathematics, as we will see in subsequent sections of this book.

When all the elements in the domain can be listed—say, $x_1, x_2, ..., x_n$—it follows that the universal quantification $\forall x P(x)$ is the same as the conjunction $P(x_1) \land P(x_2) \land \cdots \land P(x_n)$, because this conjunction is true if and only if $P(x_1), P(x_2), \ldots, P(x_n)$ are all true.


Negating Quantified Expressions

TABLE 2 De Morgan’s Laws for Quantifiers.

<table>
<thead>
<tr>
<th>Negation</th>
<th>Equivalent Statement</th>
<th>When Is Negation True?</th>
<th>When False?</th>
</tr>
</thead>
<tbody>
<tr>
<td>¬∃xP(x)</td>
<td>∀x¬P(x)</td>
<td>For every x, P(x) is false.</td>
<td>There is an x for which P(x) is true.</td>
</tr>
<tr>
<td>¬∀xP(x)</td>
<td>∃x¬P(x)</td>
<td>There is an x for which P(x) is false.</td>
<td>P(x) is true for every x.</td>
</tr>
</tbody>
</table>
Quantifiers

What are the truth values of the following statements if the domain consists of all integers:

a) $\forall n (n^2 \geq 0)$

b) $\exists n (n^2 = 2)$

c) $\forall n (n^2 \geq n)$

d) $\exists n (n^2 < 0)$
## Nested Quantifiers

<table>
<thead>
<tr>
<th>Statement</th>
<th>When True?</th>
<th>When False?</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\forall x \forall y P(x, y)$</td>
<td>$P(x, y)$ is true for every pair $x, y$.</td>
<td>There is a pair $x, y$ for which $P(x, y)$ is false.</td>
</tr>
<tr>
<td>$\forall y \forall x P(x, y)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\forall x \exists y P(x, y)$</td>
<td>For every $x$ there is a $y$ for which $P(x, y)$ is true.</td>
<td>There is an $x$ such that $P(x, y)$ is false for every $y$.</td>
</tr>
<tr>
<td>$\exists x \forall y P(x, y)$</td>
<td>There is an $x$ for which $P(x, y)$ is true for every $y$.</td>
<td>For every $x$ there is a $y$ for which $P(x, y)$ is false.</td>
</tr>
<tr>
<td>$\exists x \exists y P(x, y)$</td>
<td>There is a pair $x, y$ for which $P(x, y)$ is true.</td>
<td>$P(x, y)$ is false for every pair $x, y$.</td>
</tr>
<tr>
<td>$\exists y \exists x P(x, y)$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Negating Nested Quantifiers

\[ \neg \forall x \exists y \exists z \forall w P(x,y,z,w) \]

\[ \exists x \forall y \forall z \exists w \neg P(x,y,z,w) \]
Nested Quantifiers

What are the truth values of the following statements if the domain consists of all integers:

a) \( \forall n \exists m (n^2 < m) \)

b) \( \exists n \forall m (n < m^2) \)

c) \( \forall n \exists m (n + m = 0) \)

d) \( \exists n \forall m (nm = m) \)
Back to Sets
Let $A$ and $B$ be two sets.

A is a subset of $B$, denoted by $A \subseteq B$, if the following quantified expression is true:

$$\forall x (x \in A \rightarrow x \in B)$$
Proper Subsets

❖ Let A and B be two sets.

❖ A is a proper subset of B, denoted by $A \subset B$, if the following quantified expression is true:

$$\forall x (x \in A \rightarrow x \in B) \land \exists x (x \in B \land x \notin A)$$
Let $A$ and $B$ be two sets.

$A$ and $B$ are equal, denoted by $A = B$, if the following quantified expression is true:

$$\forall x (x \in A \leftrightarrow x \in B)$$
Power Sets

❖ The power set of set $A$, denoted by $P(A)$ or $2^A$, is the set of all subsets of $A$.

❖ What is the power set of $\{1,2,3\}$? Of $\emptyset$?
Set Operations

- **Union:** \( A \cup B = \{ x \mid x \in A \lor x \in B \} \)
- **Intersection:** \( A \cap B = \{ x \mid x \in A \land x \in B \} \)
- **Difference:** \( A \setminus B = \{ x \mid x \in A \land x \notin B \} \)
- **Complement:** \( \overline{A} = \{ x \in U \mid x \notin A \} \)
- **Cartesian product:** \( A \times B = \{ (x, y) \mid x \in A \land y \in B \} \)
## Set Identities

<table>
<thead>
<tr>
<th>Identity</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A \cap U = A$</td>
<td>Identity laws</td>
</tr>
<tr>
<td>$A \cup \emptyset = A$</td>
<td></td>
</tr>
<tr>
<td>$A \cup U = U$</td>
<td>Domination laws</td>
</tr>
<tr>
<td>$A \cap \emptyset = \emptyset$</td>
<td></td>
</tr>
<tr>
<td>$A \cup A = A$</td>
<td>Idempotent laws</td>
</tr>
<tr>
<td>$A \cap A = A$</td>
<td></td>
</tr>
<tr>
<td>$\overline{(A)} = A$</td>
<td>Complementation law</td>
</tr>
<tr>
<td>$A \cup B = B \cup A$</td>
<td>Commutative laws</td>
</tr>
<tr>
<td>$A \cap B = B \cap A$</td>
<td></td>
</tr>
<tr>
<td>$A \cup (B \cup C) = (A \cup B) \cup C$</td>
<td>Associative laws</td>
</tr>
<tr>
<td>$A \cap (B \cap C) = (A \cap B) \cap C$</td>
<td></td>
</tr>
<tr>
<td>$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$</td>
<td>Distributive laws</td>
</tr>
<tr>
<td>$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$</td>
<td></td>
</tr>
<tr>
<td>$\overline{A \cap \overline{B}} = \overline{A} \cup \overline{B}$</td>
<td>De Morgan’s laws</td>
</tr>
<tr>
<td>$\overline{A \cup B} = \overline{A} \cap \overline{B}$</td>
<td></td>
</tr>
<tr>
<td>$A \cup (A \cap B) = A$</td>
<td>Absorption laws</td>
</tr>
<tr>
<td>$A \cap (A \cup B) = A$</td>
<td></td>
</tr>
<tr>
<td>$A \cup \overline{A} = U$</td>
<td>Complement laws</td>
</tr>
<tr>
<td>$A \cap \overline{A} = \emptyset$</td>
<td></td>
</tr>
</tbody>
</table>
Questions?