COMP 182 Algorithmic Thinking

Sequences and Summations

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Reading Material

- Chapter 2, Section 4

Sequences

- A sequence is a function from a subset of the set of integers (typically the subset {1,2,3,…} or {0,1,2,…}) to a set S.
- The notation a_n is used to denote the n^{th} term of the sequence.
- The sequence itself is denoted by $\{a_n\}$.

Sequences

- Examples of sequences $\{a_n\}$:
  - $a_n=1$ (the sequence is $1,1,1,1,\ldots$)
  - $a_n=n$ (the sequence is $1,2,3,4,\ldots$)
  - $a_n=1/n^2$ (the sequence is $1,1/4,1/9,1/16,\ldots$)

Sequences

A <u>*geometric progression*</u> is a sequence of the form

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

where the *initial term* a and the *common ratio* r are real numbers.

An <u>*arithmetic progression*</u> is a sequence of the form

$$a, a+d, a+2d, \ldots, a+nd, \ldots$$

where the *initial term* a and the *common difference* d are real numbers.

Sequences

- The harmonic sequence is given by

$$a_n = \frac{1}{n} \quad \text{for } n = 1, 2, 3, \ldots$$

Recurrence Relations

- A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.
- A sequence is a solution of a recurrence relation if its terms satisfy the recurrence relation.

Recurrence Relations

- Recurrence: a_0=2, and $a_n=2a_{n-1}$ for n=1,2,3,…
- Solution: 2,4,8,16,32,…
- Solution: a_n=? for n=0,1,2,…

Summations

- The sum of terms of a sequence plays an important role in counting (which is necessary for running time analysis, probability,…).
- We can consider the sum of all terms of a sequence or the sum of certain terms.

$$\sum_{n=0}^{\infty} a_n \qquad \sum_{n=i}^{n=j} a_n \qquad \sum_{n\in S} a_n$$

Sequences and Series

- The sum of a sequence yields a series S_n as follows:

$$S_n = \sum_{i=1}^{n} a_i \quad \text{for } n = 1, 2, 3, \ldots$$

- And, if a_n is defined for $n=0$:

$$S_n = \sum_{i=0}^{n} a_i \quad \text{for } n = 0, 1, 2, \ldots$$

Arithmetic Sequence and Series

$$a, a + d, a + 2d, \ldots, a + nd, \ldots$$

$$S_n = \sum_{j=0}^{n} a + dj = (n + 1)a + \frac{n(n + 1)}{2} d$$

Telescoping Sums

$$
\begin{aligned}
S_n &= \sum_{j=1}^{n}(a_j - a_{j-1}) \\
&= (a_1 - a_0) + (a_2 - a_1) + \cdots + (a_{n-1} - a_{n-2}) + (a_n - a_{n-1}) \\
&= a_n - a_0
\end{aligned}
$$

Telescoping Sums

- Find, using a telescoping sum, a closed-form formula for the summation

$$\sum_{k=1}^{n} k$$

The Perturbation Method

- Sometimes, it helps to “perturb” the sum to create another another summation and consider the difference between them.
  - You want the difference to cancel out a majority of the terms so as you get a simple equation!

The Perturbation Method

- Suppose we want to find a closed-form formula for the sum

$$1 + r + r^2 + \cdots + r^n$$

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$$rS = r + r^2 + \cdots + r^n + r^{n+1}$$

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- Solve to get an expression for S

$$\begin{aligned} S(r-1) &= r^{n+1} - 1 \\ \Rightarrow \quad S &= \frac{r^{n+1}-1}{r-1} \end{aligned}$$

The Perturbation Method

- Suppose we want to find a closed-form formula for the following sum when $r \neq 1$

$$S_n = \sum_{j=0}^{n} ar^j$$

The Perturbation Method

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$$S_n = \frac{a(r^{n+1} - 1)}{r - 1}$$

Infinite Series

- You can use results from your favorite subject: Calculus!

Infinite Series

- We seek a closed-form formula for the sum

$$\sum_{i=0}^{\infty} x^i \qquad |x| < 1$$

Infinite Series

- This is equivalent to

$$\sum_{i=0}^{\infty} x^i = \lim_{n\to\infty} \sum_{i=0}^{n} x^i$$

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when |x| < 1

Summations

Sum	***Closed Form***
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x\rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x\rvert < 1$	$\frac{1}{(1-x)^2}$

Questions?